

# Hurwitz numbers and BKP hierarchy

S.M. Natanzon\*

A. Yu. Orlov†

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## Abstract

We consider special series in ratios of the Schur functions which are defined by integers  $F \geq 0$  and  $E \leq 2$ , and also by the set of  $3k$  parameters  $n_i, q_i, t_i$ ,  $i = 1, \dots, k$ . These series may be presented in form of matrix integrals. In case  $k = 0$  these series generates Hurwitz numbers for the  $d$ -fold branched covering of connected surfaces with a given Euler characteristic  $E$  and arbitrary profiles at  $F$  ramification points. If  $k > 0$  they generate weighted sums of the Hurwitz numbers with additional ramification points which are distributed between color groups indexed by  $i = 1, \dots, k$ , the weights being written in terms of parameters  $n_i, q_i, t_i$ . By specifying the parameters we get sums of all Hurwitz numbers with  $F$  arbitrary fixed profiles and the additional profiles provided the following condition: both, the sum of profile lengths and the number of ramification points in each color group are given numbers. In case  $E = F = 1, 2$  the series may be identified with BKP tau functions of Kac and van de Leur of a special type called hypergeometric tau functions. Sums of Hurwitz numbers for  $d$ -fold branched coverings of  $\mathbb{RP}^2$  are related to the one-component BKP hierarchy. We also present links between sums of Hurwitz numbers and one-matrix model of the fat graphs.

**Key words:** Hurwitz numbers, tau functions, matrix integrals, BKP, multi-component KP, non-orientable surfaces, projective plane, Schur functions, hypergeometric functions, random partitions

## 1 Introduction

In the beautiful paper [1], A. Okounkov studied ramified coverings of the Riemann sphere with arbitrary ramification type over 0 and  $\infty$ , and simple ramifications elsewhere, and it was proved that the generating function for the related Hurwitz numbers (numbers of nonequivalent coverings with given ramification type) is a tau -function for the Toda lattice hierarchy. In further works [15–19] other examples of tau-functions for 2D Toda and KP hierarchy generating Hurwitz numbers of the sphere were constructed. In recent work [45] there was considered more general TL tau function which gives some new examples of what were called composite signed Hurwitz numbers and includes previous examples.

In the present paper we consider Hurwitz numbers of the projective plane  $\mathbb{RP}^2$ . For our purpose we use the BKP hierarchy of integrable equations introduced by V.Kac and J. van de Leur in [21]. We also present the most general weighted combinations of Hurwitz numbers for the sphere  $\mathbb{CP}^1$  which may be related to the two-component KP hierarchy which generalizes results of [45]. (A brief explanation what happens when we change the hierarchy is given in the next paragraph.) Our main result is that the BKP tau function (70) is the generating function for the certain linear combinations of Hurwitz numbers of the projective plane, see Theorem 1 and relations (67), (69).

For readers familiar with the topic, let us briefly explain the difference between TL and BKP tau functions in the context of generating of Hurwitz numbers and topology of the base.

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\*National Research University Higher School of Economics, Moscow, Russia; Institute for Theoretical and Experimental Physics, Moscow, Russia; Laboratory of Quantum Topology, Chelyabinsk State University, Chelyabinsk, Russia; email: natanzons@mail.ru

†Institute of Oceanology, Nahimovskii Prospekt 36, Moscow 117997, Russia, and National Research University Higher School of Economics, International Laboratory of Representation Theory and Mathematical Physics, 20 Myasnitskaya Ulitsa, Moscow 101000, Russia, email: orlovs@ocean.ru

The Frobenius formula for the Hurwitz numbers enumerating  $d$ -fold branched coverings of Riemann or Klein surfaces contains the sum over irreducible representations  $\lambda$  of the symmetric group (see [10–14, 20])

$$H_\Omega(\Delta^{(1)} \dots, \Delta^{(F)}) = d! \sum_{\chi} \left( \prod_{i=1}^F |C_{\Delta^{(i)}}| \frac{\chi(\Delta^{(i)})}{\chi(1)} \right) \left( \frac{\chi(1)}{d!} \right)^E, \quad (1)$$

where  $E$  is the Euler characteristic of  $\Omega$ ,  $\{\Delta^{(i)}\}$ ,  $\Delta^{(i)}$  are profiles over ramification points on  $\Omega$ ,  $\chi(\Delta)$  is a character of the symmetric group  $S_d$  evaluated at a cycle type  $\Delta$ , and  $\chi$  ranges over the irreducible complex characters of  $S_d$ . Each profile  $\Delta^{(i)}$  is a partition of  $d$  - the set of non-negative non-increasing numbers  $(d_1^{(i)}, d_2^{(i)}, \dots)$ , which describes the ramification over the point number  $i$  on the base. The weights of all partitions involved in (1) are equal:  $|\Delta^{(i)}| = d$ . The number  $|C_\Delta|$  is the number of elements in the cycle class  $\Delta$  in  $S_d$ .

The formula (1) was derived for connected  $\Omega$  which in particular implies that  $E \leq 2$ . The geometrical meaning of the formula (1) in case  $E > 2$  is unclear. We shall use the notation  $H^E$  instead of  $H_\Omega$  to denote the left hand side of (1) where we allow  $E$  to be any integer.

The Hurwitz numbers form a topological field theory [2]. They are used in mathematical physics (for instance in [2]) and in algebraic geometry [20].

Here and below we write  $\chi_\lambda(1)$  having in mind the evaluation of the irreducible character of the symmetric group  $\chi_\lambda$  at the unity element in the symmetric group, which is given by the partition  $(1^d)$ , then,  $\chi_\lambda(1) = \dim \lambda$ .

It is well-known that Schur functions  $s_\lambda$  and characters of the symmetric group  $\chi_\lambda$  are linearly dependent [30]. Soliton theory provides various series of products of the Schur functions over partitions for tau functions of various hierarchies of integrable equations. In [1] Okounkov introduced and studied the following sum

$$\sum_{\lambda} e^{\beta f(\lambda, n)} s_{\lambda}(\mathbf{p}) s_{\lambda}(\bar{\mathbf{p}}) \quad (2)$$

with

$$f(\lambda, n) = n|\lambda| + |C_\Gamma| \frac{\chi_\lambda(\Gamma)}{\chi_\lambda(1)} \quad (3)$$

where both, the partition  $\Gamma = \Gamma_d := (1^{d-2}2)$  and the number  $|C_\Gamma|$  depends only on  $d$ , see (15) below. He shown that series (2) is a tau function of the Toda lattice where power sum variables  $\mathbf{p} = (p_1, p_2, \dots)$  and  $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_1, \dots)$  together with the integer  $n$  play the role of higher times.

Moreover, (2) generates a certain class of the Hurwitz numbers (1). This class describes coverings of the Riemann sphere ( $E = 2$ ) with arbitrary given profiles  $\mathbf{p}, \bar{\mathbf{p}}$  over two given points (say, 0 and  $\infty$ ) and any number of simple ramifications described by the Young diagram  $\Gamma$  (here and below we shall omit the dependence of  $\Gamma$  on  $d$ ). One can recognize it thanks to the relation between the Schur functions and the characters of the symmetric group [30] which we re-write in a suitable form as follows:

$$s_{\lambda}(\mathbf{p}) = p_1^d \left( 1 + p_1^{-d} \sum_{\Delta \neq 1^d} |C_\Delta| \frac{\chi_{\lambda}(\Delta)}{\chi_{\lambda}(1)} \mathbf{p}_\Delta \right) \frac{\chi_{\lambda}(1)}{d!} \quad (4)$$

where the summation ranges over all partitions  $\Delta = (d_1, d_2, \dots)$  of the number  $d = |\lambda|$ , and where  $\mathbf{p}_\Delta$  is the product  $p_{d_1} p_{d_2} \dots$ . The numbers  $|C_\Delta|$  depend only on  $\Delta$  (see [30] or (15) below).

Now, it is clear from (4) that the formula (2) is a generating function for Hurwitz numbers (1) where  $\beta$ ,  $\mathbf{p}$  and  $\bar{\mathbf{p}}$  are formal parameters. Basically, the Taylor coefficients in the terms  $p_\Delta \bar{p}_{\Delta'} \beta^b$ , up to a factor, coincide with the number of covers with the ramification type  $\Delta, \Delta'$  over two points and further of type  $\Gamma = 1^{|\Delta|-2}2$  over  $b$  points.

We present a larger class of series of type (2), which we shall call generating Hurwitz series:

$$\tau^{(E, F)} \left( N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F)} | \{q_i, t_i, n_i\}, \beta \right) = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_{\lambda}^{\mathbf{q}, \mathbf{t}, \mathbf{n}, \beta}(n) (\chi_{\lambda}(1))^{E-F} \prod_{j=1}^F s_{\lambda}(\mathbf{p}^{(j)}), \quad E \in \mathbb{Z} \quad (5)$$

where  $\mathbf{p}^{(j)} = (p_1^{(j)}, p_2^{(j)}, \dots)$  are power sums variables,  $a_i, q_i, n_i \in \mathbb{C}$ , and

$$r_\lambda^{\mathbf{q}, \mathbf{t}, \mathbf{n}, \beta}(n) = e^{\beta f(\lambda, \mathbf{n})} \prod_{i=1}^k \left( \frac{s_\lambda(\mathbf{p}(q_i t_i^n, t_i))}{s_\lambda(\mathbf{p}(0, t_i))} \right)^{n_i} \quad (6)$$

while power sum variables  $\mathbf{p}(q_i, t_i) = (p_1(q_i, t_i), p_2(q_i, t_i), \dots)$  are specified as follows

$$p_m(q_i, t_i) := \frac{1 - q_i^m}{1 - t_i^m}, \quad p_m(0, t_i) := \frac{1}{1 - t_i^m} \quad (7)$$

On the relation of these  $\mathbf{p}(q_i, t_i)$  to Macdonald polynomials see the Appendix A.1. We have

$$\left( \frac{s_\lambda(\mathbf{p}(q_i, t_i))}{s_\lambda(\mathbf{p}(0, t_i))} \right)^{n_i} = (1 - q_i)^{n_i d} \left( \frac{1 + \sum_{\Delta \neq 1^d} |C_\Delta| \frac{\chi_\lambda(\Delta)}{\chi_\lambda(1)} w_\Delta(q_i, t_i)}{1 + \sum_{\Delta \neq 1^d} |C_\Delta| \frac{\chi_\lambda(\Delta)}{\chi_\lambda(1)} w_\Delta(0, t_i)} \right)^{n_i} \quad (8)$$

where

$$w_\Delta(q_i, t_i) = \left( \frac{1 - t_i}{1 - q_i} \right)^d \prod_{j=1}^\ell \frac{1 - q_i^{d_j}}{1 - t_i^{d_j}}, \quad \Delta = (d_1, \dots, d_\ell) \quad (9)$$

For the sake of brevity we shall also write  $\tau_r^{(\mathbf{E}, \mathbf{F})}$  instead of  $\tau^{(\mathbf{E}, \mathbf{F})}(N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(\mathbf{F})} | \{q_i, t_i, n_i\}, \beta)$ .

The series (5) are interesting because of three aspects.

First, for  $\mathbf{F} = \mathbf{E} = 1, 2$  they may be related to the  $\mathbf{E}$ -component BKP hierarchy of integrable equations. For  $\mathbf{E} = 2$ , they also may be related to the 2-component KP hierarchy (2KP) and Toda lattice hierarchy. The cases  $\mathbf{E} = 1$  (one-component BKP) and  $\mathbf{E} = 2$  (2-component KP, and also 2-component BKP) describes specializations of tau functions of hypergeometric type studied respectively in [31] and in [25], [26] (in case  $n_i = \pm 1$ , such specialization was introduced in [26], [29] in the context of hypergeometric functions of matrix argument [27] and Milne's hypergeometric functions [28]).

Second, by (4) series (5) may be viewed as a generating function for certain weighted combinations of the right hand sides of eq. (1),  $H^\mathbf{E}$ . To see it, again, we use (4) replacing each Schur function by its character expansion. One can see that thanks to (4) we obtain linear combinations of terms  $H^\mathbf{E}$ . We notice that the factor  $r_\lambda$  in (5) contains only ratios of the Schur functions and due to (4) does not contribute to the power of  $\chi_\lambda(1)$  in (1), this power is the result of the multiplication of the  $\mathbf{E}$  Schur functions in the left hand side of (5).

At last, series (5) may be obtained as integrals of 2KP and BKP tau functions over matrices.

Let us note that series (5) where  $\mathbf{E} = 2$  and  $n_i = 1$ ,  $q_i = t_i^{a_i} \rightarrow 1$ ,  $i = 1, \dots, k$  we introduces in [16], and the case  $\mathbf{E} = 2$  and  $n_i = \pm 1$ ,  $q_i = t_i^{a_i} \rightarrow 1$ ,  $i = 1, \dots, k$  was studied in [45].

The generating Hurwitz series labeled by  $\mathbf{E} - 1, \mathbf{F} - 1$  may be obtained from the series labeled by  $\mathbf{E}, \mathbf{F}$  as follows

$$\tau^{(\mathbf{E}-1, \mathbf{F}-1)}(N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(\mathbf{E}-1)}) = \left[ e^{L^{(\mathbf{E})}} \cdot \tau^{(\mathbf{E}, \mathbf{F})}(N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(\mathbf{E})}) \right]_{\mathbf{p}^{(\mathbf{E})}=0} \quad (10)$$

where  $L^{(\mathbf{E})}$  is the Laplace operator  $\sum_{m>0} \left( \frac{m}{2} \frac{\partial^2}{\partial p_m^2} + \frac{\partial}{\partial p_{2m-1}} \right)$ , where each  $p_m$  is  $p_m^{(\mathbf{E})}$ , see Section 5.

Now for  $\mathbf{E} = \mathbf{F} = 2$  we have the following series

$$\tau^{2KP}(n, \mathbf{p}, \bar{\mathbf{p}}) = \sum_{\lambda} r_\lambda(n) s_\lambda(\mathbf{p}) s_\lambda(\bar{\mathbf{p}}) \quad (11)$$

which may be identified with TL or, the same, 2KP tau function, see Section 3 below.

As a result one may conclude that, similar to the Okounkov tau function, the TL tau function (11) generates linear combinations of number of covering of the sphere. These specific combinations of Hurwitz numbers ("composite signed Hurwitz numbers") in case  $q_i \rightarrow 1$  are written down in [45] and for our convenience are also reproduced in the present text.

Put  $\mathbf{E} = \mathbf{F} = 1$ . The series

$$\tau^{BKP}(n, \mathbf{p}) = \sum_{\lambda} r_\lambda(n) s_\lambda(\mathbf{p}) \quad (12)$$

with the same  $r_\lambda(n)$  is also a tau function [31] where the set  $\mathbf{p}$  plays the role of higher times, but now it is a different hierarchy, namely, the BKP hierarchy introduced by Kac and van de Leur [21]. This case will be considered in the present paper. It is easy to see that now  $E = 1$ . Thus, according to Frobenius formula (1), function (12) is a generating function for Hurwitz numbers of the projective plane  $\mathbb{RP}^2$ .

If we consider the further case  $E = 0$  (the coverings of the elliptic curve) with the help of the series

$$\tau = \sum_{n, \lambda} r_\lambda(n) \quad (13)$$

then we see it is not a tau function as there are no time variables here. This expression may be related to the trace of the certain diagonal  $\hat{G}L_\infty$  element in the fermionic Fock space.

Thus, on the formal level we can explain the appearance of different hierarchies of integrable equations in the description of Hurwitz counting problem.

This paper is the detalization of the consideration above. We shall study only the case all  $q_i \rightarrow 1$ . The paper is organized as follows. In Section 2 we describe Hurwitz numbers and its combinations generating solutions of integrable systems. In Section 3 we recall some facts about BKP and TL hierarchies. We need a special class of tau functions which we call hypergeometric tau functions. In Section 4.1 in particular we review TL tau functions generating composite signed Hurwitz numbers according to [45]. However we need a modification caused by semiinfinity of TL which we need to compare results with the BKP case later in Section 5. In sections 4 and 5 we construct Hurwitz  $\tau$ -functions for BKP hierarchy and find its connection with Hurwitz  $\tau$ -functions for the semiinfinte 2DToda hierarchy.

In Section 6 we present a relation between fat graph counting obtained from the one-matrix models and sums of Hurwitz numbers, and also write down an analogue of the one-matrix model which generates similar sums for Hurwitz numbers of the projective plane. We show ways to get Hurwitz generating series (5) in form of matrix integrals. These are integrals of the simplest 2KP and BKP functions whose as functions of products of matrices. One of the way to diminish the Euler  $E$  of the generating series by 1 is to replace one of 2KP tau functions in the integrand by a BKP tau function.

In the Appendix B we discuss Hurwitz generating series in the context of matrix integrals. In the Appendix A.3 we write down the fermionic expression of the Hurwitz generating series (5) where  $E=F > 2$ .

## 2 Weighted sums of Hurwitz numbers

### 2.1 Hurwitz numbers

For a partition  $\Delta$  of a number  $d = |\Delta|$  denote by  $\ell(\Delta)$  the number of the non-vanishing parts. For the Young diagram, corresponding to  $\Delta$ , the number  $|\Delta|$  is the weight of the diagram and  $\ell(\Delta)$  is the number of rows. Denote by  $(d_1, \dots, d_\ell)$  the Young diagram with rows of length  $d_1, \dots, d_\ell$  and corresponding partition of  $\sum d_i$ .

Hurwitz number  $H_\Omega(d, \Delta^{(1)}, \dots, \Delta^{(F)})$  is defined by a connected surface  $\Omega$  and partitions  $\Delta^{(1)}, \dots, \Delta^{(F)}$  of the number  $d = |\Delta^{(i)}|$ ,  $i = 1, \dots, F$ . The Hurwitz number  $H_\Omega(d, \Delta^{(1)}, \dots, \Delta^{(F)})$  is the weighted number of branched coverings of the surface  $\Omega$  by other surfaces (connected or non-connected) with fixed critical values  $z_1, \dots, z_F \in \Omega$  of topological types  $\Delta^{(1)}, \dots, \Delta^{(F)}$ . More precisely,  $z \in \Omega$  is the critical value of the branched covering  $f : \Sigma \rightarrow \Omega$  if  $z = f(p)$ , where  $p \in \Sigma$  is a critical point of  $f$ . Consider degrees  $d_1, \dots, d_\ell$  of  $f$  in all preimages  $f^{-1}(z)$ . The partition  $(d_1, \dots, d_\ell)$  of  $d = \deg(f)$  is called the topological type of the critical value  $z$ . We say that branched coverings  $f' : \Sigma' \rightarrow \Omega$  and  $f'' : \Sigma'' \rightarrow \Omega$  are the same, if there exists a homeomorphism  $g : \Sigma' \rightarrow \Sigma''$  such that  $f' = f''g$ . Then

$$H_\Omega(d, \Delta^{(1)}, \dots, \Delta^{(F)}) = \sum \frac{1}{|\text{Aut}(f)|} \quad , \quad (14)$$

where the sum is taken over all branched coverings  $f$  of  $\Omega$ , with the critical values  $z_1, \dots, z_F \in \Omega$  of the topological types  $\Delta^{(1)}, \dots, \Delta^{(F)}$  respectively. This number is independent of the positions of the branching points  $z_i$ .

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. They are well studied for orientable  $\Omega$ . In this case the Hurwitz number coincides with the weighted number of holomorphic branched coverings of a Riemann surface  $\Omega$  by another Riemann

surfaces, having critical values  $z_1, \dots, z_F \in \Omega$  of topological types  $\Delta^{(1)}, \dots, \Delta^{(F)}$  respectively. The well known isomorphism between Riemann surfaces and complex algebraic curves gives the interpretation of the Hurwitz numbers as the numbers of morphisms of complex algebraic curves.

In this work we consider the Hurwitz numbers for non-orientable  $\Omega$  without boundary. They have also two other interpretations: as the numbers of the branched coverings of a Klein surface without boundary by another Klein surface, and as the number of morphisms of real algebraic curves without real points. Klein surfaces are factors of Riemann surfaces by antiholomorphic involutions. They correspond to real algebraic curves. Real points of real curves correspond to fixed points of the involutions and boundary points of the Klein surfaces (see [3–5]). In this paper we consider only surfaces without boundaries. But an analog of the Hurwitz numbers for surfaces with boundaries also exists ([6, 7]).

The Hurwitz numbers are closely connected with irreducible representations of  $S_d$ . The action of any permutation  $s \in S_d$  split the set  $1, \dots, d$  on subsets cardinality  $(d_1, \dots, d_\ell)$  and thus generate a partition  $\Delta(s) = (d_1, \dots, d_\ell)$  of  $d$ . This partition is called as cyclic type of  $s$ . Conversely, any partition  $\Delta$  of  $d$  generate the set  $C_\Delta \subset S_d$ , consisted of permutation of cyclic type  $\Delta$ . The cardinality of  $C_\Delta$  is equal to

$$|C_\Delta| = \frac{|\Delta|!}{z_\Delta}, \quad z_\Delta = \prod_{i=1}^{\infty} i^{m_i} m_i! \quad (15)$$

where  $m_i$  denotes the number of parts equal to  $i$  of the partition  $\Delta$  (then a partition  $\Delta$  is often denoted by  $1^{m_1} 2^{m_2} \dots$ ).

Moreover, if  $s_1, s_2 \in C_\Delta$ , then  $\chi(s_1) = \chi(s_2)$  for any complex characters  $\chi$  of  $S_d$ . Thus we can define  $\chi(\Delta)$  for a partition  $\Delta$ , as  $\chi(\Delta) = \chi(s)$  for  $s \in C_\Delta$ .

The Frobenius formula [10–14, 20] says that

$$H_\Omega(d, \Delta^{(1)}, \dots, \Delta^{(F)}) = d! \sum_{\chi} \left( \prod_{i=1}^F |C_{\Delta^{(i)}}| \frac{\chi(\Delta^{(i)})}{\chi(1)} \right) \left( \frac{\chi(1)}{d!} \right)^E, \quad (16)$$

where  $E$  is the Euler characteristic of  $\Omega$  and  $\chi$  ranges over the irreducible complex characters of  $S_d$ , associated with Young diagrams of wight  $d$ .

In what follows we shall construct the generating series for the numbers

$$H_{d,N}^{E,F}(\Delta^{(1)}, \dots, \Delta^{(F)}) := d! \sum_{|\lambda|=d, \ell(\lambda) \leq N} \left( \prod_{i=1}^F |C_{\Delta^{(i)}}| \frac{\chi_\lambda(\Delta^{(i)})}{\chi_\lambda(1)} \right) \left( \frac{\chi_\lambda(1)}{d!} \right)^E, \quad (17)$$

depending on free integer parameters  $E, F, d, N$ . For  $N \geq d$  the number  $H_{d,N}^{E,F}$  does not depend on  $N$ . In particular for  $N \geq d$  we have  $H_{d,N}^{2,F}(\Delta^{(1)}, \dots, \Delta^{(F)}) = H_{\mathbb{CP}^1}(d, \Delta^{(1)}, \dots, \Delta^{(F)})$  and  $H_{d,N}^{1,F}(\Delta^{(1)}, \dots, \Delta^{(F)}) = H_{\mathbb{RP}^2}(d, \Delta^{(1)}, \dots, \Delta^{(F)})$ .

**Suitable notations: occupation numbers.** Given  $d$  we have a finite number, say denoted by  $d^* + 1$ , of all different partitions of  $d$ ,  $\Delta_0, \dots, \Delta_{d^*}$  (the value of  $d^* = d^*(d)$  is unimportant for us). For a given  $d$  it is convenient to enumerate partitions by a label, say,  $j = 0, 1, \dots, d^*$  where we reserve  $j = 0$  for the partition  $1^d$ :  $\Delta_0 = 1^d$ . By  $c_j \geq 0$  denote how many times a partition  $\Delta_j$  occurs in the set of arguments of  $H_{d,N}^E$  and write

$$H_\Omega(d; \underbrace{\Delta_0, \dots, \Delta_0}_{c_0}, \dots, \underbrace{\Delta_{d^*}, \dots, \Delta_{d^*}}_{c_{d^*}}) =: H_\Omega(d; \mathbf{c})$$

where  $\mathbf{c} = \mathbf{c}(d) = (c_0, \dots, c_{d^*-1})$ . By analogue with particle and statistical physics we will call  $c_j$  the occupation number of the state  $j$  (similar notations were used in [9]). As one concludes from (16) the Hurwitz numbers do not depend on  $c_0$ , and it is obvious from the geometrical point of view since it is related to the absence of branching.

We shall also use the mixed notations, where we split the arguments of  $H_\Omega$  into two groups:

$$H_\Omega(d; \mathbf{c}, \Delta^{(1)}, \dots, \Delta^{(F)}) := H_\Omega \left( d; \underbrace{\Delta_0, \dots, \Delta_0}_{c_0}, \dots, \underbrace{\Delta_{d^*}, \dots, \Delta_{d^*}}_{c_{d^*}}, \Delta^{(1)}, \dots, \Delta^{(F)} \right)$$

We shall consider weighted sums of Hurwitz numbers over the occupation numbers keeping a group of  $F$  partitions fixed. The series (5) yields examples of such sums, which have the following form

$$\tau^{(E,F)} \left( N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F)} | \{q_i, t_i, n_i\}, \beta \right) := \sum_d \sum_{\mathbf{c}} \omega(d, \mathbf{c}) H_{d,N}^{E,F+|\mathbf{c}|}(\mathbf{c}, \Delta^{(1)}, \dots, \Delta^{(F)}) \prod_{i=1}^F \mathbf{p}_{\Delta^{(i)}}^{(i)}$$

where the weights  $\omega$  depend on  $n, \mathbf{p}^{(j)}, \beta$  and on  $q_i, t_i, n_i, i = 1, \dots, k$ , and will be found in the next section.

## 2.2 Weighted sums of Hurwitz numbers.

Our goal is to explain what kind of information related to Hurwitz number is hidden in series (5) depending on  $3k$  parameters  $q_i, t_i, n_i$ .

**The case**  $k = 0, b = 0$ . First of all we notice that if the factor  $r_\lambda$  is equal to 1, the series

$$\tau_1^{E,F}(N, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F)}) = \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} (s_\lambda(\mathbf{p}_\infty))^E \prod_{i=1}^F \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)}$$

generate Hurwitz numbers themselves. Here we use the notation  $\mathbf{p}_\infty = (1, 0, 0, \dots)$ , then, from (4) we obtain the known relation [30]

$$s_\lambda(\mathbf{p}_\infty) = \frac{\dim \lambda}{|\lambda|!} = \frac{\chi_{(1^{|\lambda|})}^\lambda}{|\lambda|!} \quad (18)$$

where  $\dim \lambda$  is the dimension of the irreducible representation  $\lambda$  of the symmetric group  $S_d$ ,  $d = |\lambda|$ .

Indeed, thanks to (4) we easily obtain

$$\tau_1^{E,F}(\mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F)}) = \sum_{\Delta^{(1)}, \dots, \Delta^{(F)}} H_{d,N}^{E,F} \left( d; \Delta^{(1)}, \dots, \Delta^{(F)} \right) \prod_{i=1}^F \mathbf{p}_{\Delta^{(i)}}^{(i)} \quad (19)$$

where  $H_{d,N}^{E,F}$  is given by (17).

**The case**  $k, b > 0$ . In case the prefactor  $r_\lambda$  is not identical to 1 the series (5) generates not the Hurwitz numbers but certain linear combinations (weighted sums) of these numbers. This is the subject of Proposition 1 below. The contribution of the exponential prefactor in (6) is already explained in [1] and we concentrate on the contribution of the ratio of the Schur functions in (6).

Let us note that we need some preliminary work to explain what sort of weighted sums we are going to obtain. These are weighted sums over additional partitions. Each additional partition belongs to one of  $2k$  colored group. The number of the partitions in each group is not fixed, the weight is defined by the partition and by the color  $i$ , where  $i = 1, \dots, 2k$ , of the partition, namely, the weight depends on the values of the complex parameters  $q_i, t_i, n_i$ .

We need additional notations for the additional partitions. <sup>1</sup> First of all we notice that the additional partitions are profiles of the branch points and all profiles has the same profile weight (the weight of the related partition) equal to  $d$ . Denote the set of all partitions of the weight  $d$  by  $\mathbf{P}_d$ . We recall:  $\{\Delta_j \in \mathbf{P}_d, j = 0, \dots, d^*\}$  is the complete set of different partitions of the weight  $d$ , and  $\Delta_0 = 1^d$ .

Let us consider a set  $D \in \mathbf{P}_d^{\times 2k}$  of partitions of the same weight  $d$  separated into  $k$  pairs of (color) groups. Color groups are numbered by  $k$  pairs of indexes  $1, \bar{1}, \dots, k, \bar{k}$ . Then each partition from this set may be indexed by a superscript which numbers the color of the partition and by a subscript which indexes its number among all partitions of a given weight  $d$ . Say  $\Delta_j^i$  ( $\Delta_j^{\bar{i}}$ ) is the partition numbered by  $j$  in the set of all partitions of  $d$  from the color group  $i$  ( $\bar{i}$ ). Then the occupation numbers  $c_j^i$  (and  $\bar{c}_j^{\bar{i}}$ ) where  $i = 1, \dots, k, j = 0, \dots, d^* - 1$  says how many times the profile  $\Delta_j$  occurs in the color group

<sup>1</sup>The notations  $\Gamma = 1^{d-2}$  and  $\Delta^{(i)}, i = 1, \dots, F$  we will keep for fixed partitions, while for additional ones we shall use subscripts and superscripts without brackets like  $\Delta_j^i$  below.

$i$  (respectively  $\bar{i}$ ). Thus  $D = D(\mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k)$  is defined by the  $2k$  sets of occupation numbers: by  $\mathbf{c}, \bar{\mathbf{c}}^i = (c_0^i, \bar{c}_0^i, \dots, c_{d^*-1}^i, \bar{c}_{d^*-1}^i)$ ,  $i = 1, \dots, k$ .

We introduce the notation  $|\mathbf{c}^i| = \sum_{j=1}^{d^*-1} c_j^i$  (and  $|\bar{\mathbf{c}}^i| = \sum_{j=1}^{d^*-1} \bar{c}_j^i$ ) which says how many partitions different from  $\Delta_0$  is contained in the color group  $i$  (resp.  $\bar{i}$ ) of  $D(\mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k)$ . The set  $D$  we call the set of additional partitions. And  $\mathbf{C} := \sum_{i=1}^k (|\mathbf{c}^i| + |\bar{\mathbf{c}}^i|)$  denotes the total number of additional profiles (where we exclude all profiles equal to  $\Delta_0 = 1^d$ ).

We consider  $H_{d,N}^{E,F}$  (which are Hurwitz numbers if  $N \geq d$ , see (17)) as functions of the set of partitions  $\Delta^{(1)}, \dots, \Delta^{(F)}$ , also the set of  $b$  partitions  $\Gamma = 1^{d-2}2$  and of the set of the additional partitions  $D(\mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k)$ . Using the mixed notation we write

$$H_{d,N}^{E,F+b+\mathbf{C}}(\mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)} \dots, \Delta^{(F)})$$

We want to consider weighted sums of such Hurwitz numbers over occupation numbers of the additional partitions  $\mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k$ .

Next we write down the corresponding weights.

**Weight functions.** We recall that we introduced partitions  $\Delta_j^i, \Delta_j^{\bar{i}}$  and  $\bar{i} = 1, \dots, k$  which occur  $c_j^i$  (respectively  $\bar{c}_j^{\bar{i}}$ ) times in the argument of  $H_{d,N}^{E,F+b+\mathbf{C}}$  (and  $H_{d,N}^{E,F+b+\mathbf{C}}$  are the Hurwitz numbers in case  $N \geq d$ ).

In what follows we shall need the set  $i = 1, \dots, k$  of the following functions of 3 parameters  $n_i, q_i, t_i$ .

$$W_i(\mathbf{c}^i) : = W(n_i, q_i, t_i, \mathbf{c}^i) = (-1)^{|\mathbf{c}^i|} (-n_i)_{|\mathbf{c}^i|} \prod_{j=1}^{d^*} \frac{(w_{\Delta_j^i}(q_i, t_i))^{c_j^i}}{c_j^i!} \quad (20)$$

$$W_i^*(\bar{\mathbf{c}}^i) : = W(-n_i, 0, t_i, \bar{\mathbf{c}}^i) = (-1)^{|\bar{\mathbf{c}}^i|} (n_i)_{|\bar{\mathbf{c}}^i|} \prod_{j=1}^{d^*} \frac{(w_{\Delta_j^{\bar{i}}}(0, t_i))^{\bar{c}_j^{\bar{i}}}}{\bar{c}_j^{\bar{i}}!} \quad (21)$$

where  $|\mathbf{c}^i| := c_1^i + \dots + c_{d^*}^i$ ,  $|\bar{\mathbf{c}}^i| := \bar{c}_1^i + \dots + \bar{c}_{d^*}^i$ , and the notation  $(n)_m := n(n+1) \dots (n+m-1)$  serves for the Pochhammer symbol. Functions  $w_{\Delta_j^i}$  are defined as follows

$$w_{\Delta_j^i}(q_i, t_i) = \left( \frac{1-t_i}{1-q_i} \right)^d \prod_{s=1}^{\ell(\Delta_j^i)} \frac{1-q_i^{d_s^{ij}}}{1-t_i^{d_s^{ij}}}, \quad q_i, t_i \in \mathbb{C} \quad (22)$$

where  $d_1^{ij}, \dots, d_\ell^{ij}$  are parts of the partition  $\Delta_j^i$  and  $\sum_{s=1}^{\ell} d_s^{ij} = d$ .

**Remark 1.** As we see  $w_\lambda(t, q) = w_\lambda(q, t)^{-1}$ , in particular  $w_\lambda(t, t) = 1$ . Also  $w_{(1^d)}(q, t) = 1$  for each  $q, t$ .

**Remark 2.**  $w_\lambda(q^{-1}, t) = w_\lambda(q, t^{-1}) = (-1)^{d-\ell(\lambda)} w_\lambda(q, t)$

**Remark 3.** As we see  $w_\lambda(q, 1) = \delta_{\lambda, 1^d}$  for  $q \neq 1$ . It means that  $W_i^* = \delta_{|\bar{\mathbf{c}}^i|, 0}$  in case  $q \neq 1, t = 1$ .

**Remark 4.** The multiplier  $\frac{1}{c_j^i!}$  in (20) corresponds to the permutation between identical profiles labeled by the partition  $\Delta_j^i$ .

**Weighted sums.** We shall consider linear combinations of Hurwitz numbers related to different sets of partitions. To describe these sets we fix the weight of partitions, say,  $d$ . Then we have a finite number of all different partitions of this weight, we denote this number by  $1 + d^*$ . Let us enumerate partitions from this set by a subscript:  $\Delta_j$ ,  $j = 1, \dots, d^*$ . In what follows we need colored groups of partitions, the color will be labeled by a superscript. Thus we have the set  $\Delta_j^i$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, d^*$ , provided  $\Delta_1^i$ ,  $i = 1, \dots, k$ . In our notation of functions defined on such sets, say, the Hurwitz number  $H_\Omega$ , we shall write  $H_\Omega(\{c_j^i, i = 1, \dots, k, j = 1, \dots, d^*\})$  instead of  $H_\Omega(\{\Delta_j^i, i = 1, \dots, k, j = 1, \dots, d^*\})$  replacing each partition  $\Delta_j^i$  by the *occupation number*  $c_j^i \geq 0$  which shows how many times the partition  $\Delta_j^i$  occurs to be the argument of the function. For us it is important to keep in mind the colored groups, and we

shall use the following notation:  $\Delta^i = \Delta_1^i, \dots, \Delta_{d^*}^i$  and  $\mathbf{c}^i(d) = c_1^i, \dots, c_{d^*}^i$ . In what follows we shall use two independent sets of occupation numbers:  $\mathbf{c}^i(d)$  and  $\bar{\mathbf{c}}^i(d) = \bar{c}_1^i, \dots, \bar{c}_{d^*}^i$ .

We consider

$$H_{d,N}^{\mathbf{E}, \mathbf{F} + b + \mathbf{C}} \left( d; \mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbf{F})} \right) \quad (23)$$

the set of partitions  $\{\Delta_j^i, j = 1, \dots, d^*\}$  is the set of all different partitions of the weight  $d$  except  $1^d$  there are  $k$  copies of such sets, each set is labeled by the superscript  $i = 1, \dots, k$

the number  $c_j^i \geq 0$  ( $\bar{c}_j^i \geq 0$  says how many times the partition  $\Delta_j^i$  (resp.  $\Delta_j^i$ ) occurs to appear among arguments of  $H_{d,N}^{\mathbf{E}, \mathbf{F} + b + \mathbf{C}}$

where  $\Gamma_1 = \dots = \Gamma_b = 1^{d-2}$ .

Thus  $H_{d,N}^{\mathbf{E}}$  depends on the set of  $b + \mathbf{F} + \sum_{i=1}^k \sum_{j=1}^{d^*} (c_j^i + \bar{c}_j^i)$  partitions.

We will study the weighted sums of  $H_{d,N}^{\mathbf{E}}$  over the sets  $c_j^i$  and  $\bar{c}_j^i$  while partitions  $\Delta^{(1)} \dots, \Delta^{(\mathbf{F})}$  are fixed.

Denote by

$$L_{d,N}^{\mathbf{E}} \left( d | \{n_i, q_i, t_i\} | b | \Delta^{(1)}, \dots, \Delta^{(\mathbf{F})} \right) = \sum_{\substack{\mathbf{c}^1, \dots, \mathbf{c}^k \\ \bar{\mathbf{c}}^1, \dots, \bar{\mathbf{c}}^k}} H_{d,N}^{\mathbf{E}, \mathbf{F} + b + \mathbf{C}} \left( d; \mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbf{F})} \right) \prod_{i=1}^k W_i(\mathbf{c}^i) W_i^*(\bar{\mathbf{c}}^i) \quad (24)$$

where both  $W_i$  and  $W_i^*$  depend on  $\{n_i, q_i, t_i\}$  are defined by (20) and (21).

**Remark 5.** In case  $d \leq N$  and  $\mathbf{E} = 1, 2$  the number  $H_{d,N}^{\mathbf{E}}$  is the Hurwitz number. In what follows we will refer this case as Hurwitz case. The  $3k$ -parametric sum (24) involves the Hurwitz number of  $b + \mathbf{F} + \sum_{i=1}^k (|\mathbf{c}^i| + |\bar{\mathbf{c}}^i|)$  partitions where  $b + \mathbf{F}$  partitions are fixed and the summation ranges over the rest  $\mathbf{C} := \sum_{i=1}^k (|\mathbf{c}^i| + |\bar{\mathbf{c}}^i|)$  partitions.

In case we choose  $d \leq N$  we obtain weighted sums of Hurwitz numbers  $H_{\Omega} = H_{d,N}^{\mathbf{E}}$  for the base surface with Euler characteristic  $\mathbf{E}$ .

**Proposition 1.** *The series (5) generates weighted sums  $L_{d,N}^{\mathbf{E}}$ :*

$$\tau^{(\mathbf{E}, \mathbf{F})} \left( N, n, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(\mathbf{F})} | \{q_i, t_i, n_i\}, \beta \right) = \sum_{d,b=0}^{\infty} \frac{\beta^b}{b!} \sum_{\Delta^{(1)}, \dots, \Delta^{(\mathbf{F})}} L_{d,N}^{\mathbf{E}} \left( \{n_i, q_i^{a_i} t_i^{n_i}, t_i\} | b | \Delta^{(1)}, \dots, \Delta^{(\mathbf{F})} \right) \prod_{i=1}^{\mathbf{F}} \mathbf{p}_{\Delta^{(i)}}^{(i)}$$

*Proof.* Let us consider each factor in (5) related to a term labeled by  $\lambda$ .

We start with the factors in (6). First of all from (7)

$$p_m(q_i, t_i) := \frac{1 - q_i^m}{1 - t_i^m}, \quad p_m(0, t_i) := \frac{1}{1 - t_i^m}$$

for a partition  $\Delta = (d_1, d_2, \dots)$  we obtain

$$\mathbf{p}_{\Delta}(q_i, t_i) := (p_1(q_i, t_1))^{d_1} (p_2(q_i, t_1))^{d_2} \dots = \left( \frac{1 - q_i}{1 - t_i} \right)^{d_1} \left( \frac{1 - q_i^2}{1 - t_i^2} \right)^{d_2} \dots$$

Then from (4) we obtain (8) and (9). Expanding (8) we obtain

$$\sum (-1)^{|\mathbf{c}^i|} (-n_i)_{|\mathbf{c}^i|} \prod_{j=1}^{d^*} \frac{\left( w_{\Delta_j^i}(q_i, t_i) \right)^{c_j^i}}{c_j^i!} \left( \frac{|C_{\Delta_j^i}| \chi_{\lambda}(\Delta_j^i)}{\chi_{\lambda}(1)} \right)^{c_j^i}$$



in enumerator and

$$\sum (-1)^{|\bar{\mathbf{c}}^i|} (n_i)_{|\bar{\mathbf{c}}^i|} \prod_{j=1}^{d^*} \frac{\left(w_{\Delta_j^i}(0, t_i)\right)^{\bar{c}_j^i}}{\bar{c}_j^i!} \left(\frac{|C_{\Delta_j^i}| \chi_\lambda(\Delta_j^i)}{\chi_\lambda(1)}\right)^{\bar{c}_j^i}$$

in the denominator. The summation ranges over all partitions of the weight  $d = |\lambda|$ , namely, over all occupation numbers  $c_j^i$  and  $\bar{c}_j^i$ .

Next, for the factor  $e^{\beta f(\lambda, n)}$  in (6) from (3) we obtain

$$\sum_b \frac{\beta^b}{b!} \left(\frac{|C_\Gamma| \chi_\lambda(\Gamma)}{\chi_\lambda(1)}\right)^b$$

For each of  $\mathbb{F}$  Schur functions in the product (5), we use (4) to write

$$s_\lambda(\mathbf{p}^{(j)}) = \left(p_1^{(j)}\right)^d \left(1 + \left(p_1^{(j)}\right)^{-d} \sum_{\Delta \neq 1^d} |C_{\Delta^{(j)}}| \frac{\chi_\lambda(\Delta^{(j)})}{\chi_\lambda(1)} \mathbf{p}_{\Delta^j}^{(j)}\right) \frac{\chi_\lambda(1)}{d!}$$

Finely we sum the product of all mentioned factors over  $\lambda$ , and using the defying relation (1) for Hurwitz numbers we obtain the Proposition 1.  $\square$

Let us reduce  $3k$ -parametric families of weighted Hurwitz numbers as in examples below.

**Example 1.** We take  $q_i = t_i$ ,  $i = 1, \dots, k$ . (See Remark 1). Then we obtain  $2k$ -parametric family:

$$L_d^{\mathbb{E}} \left( b \mid \{n_i, t_i, t_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = \sum_{\substack{\mathbf{c}^1, \dots, \mathbf{c}^k \\ \bar{\mathbf{c}}^1, \dots, \bar{\mathbf{c}}^k}} H_\Omega \left( d; \mathbf{c}^1, \bar{\mathbf{c}}^1, \dots, \mathbf{c}^k, \bar{\mathbf{c}}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) W$$

$$W = \prod_{i=1}^k \frac{(-n_i)_{|\mathbf{c}^i|} (n_i)_{|\bar{\mathbf{c}}^i|}}{(-1)^{|\mathbf{c}^i| + |\bar{\mathbf{c}}^i|}} \prod_{j=1}^{d^*} \frac{1}{c_j^i!} \prod_{j=1}^{d^*} \frac{\left(w_{\Delta_j^i}(0, t_i)\right)^{\bar{c}_j^i}}{\bar{c}_j^i!}$$

Take further  $n_i = -1$ ,  $i = 1, \dots, k$ . Due to the factor  $(n_i)_{|\bar{\mathbf{c}}^i|}$  there is a single profile in each color group labeled by bar. After some changing of notations under summation we obtain

$$\sum_{\substack{\mathbf{c}^1, \dots, \mathbf{c}^k \\ \Delta^1, \dots, \Delta^k}} H_\Omega \left( d; \mathbf{c}^1, \Delta^1, \dots, \mathbf{c}^k, \Delta^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) W \quad (25)$$

$$W = - \left( \prod_{i=1}^k \frac{|\mathbf{c}^i|!}{(-1)^{|\mathbf{c}^i|}} \prod_{j=1}^{d^*} \frac{1}{c_j^i!} \right) \prod_{i=1}^k w_{\Delta^i}(0, t_i) \quad (26)$$

where

$$w_{\Delta^i}(0, t_i) = (1 - t_i)^d \prod_{s=1}^{\ell(\Delta^i)} \frac{1}{1 - t_i^{d_s^i}}, \quad (27)$$

where  $\Delta^i = (d_1^i, d_2^i, \dots)$ ,  $i = 1, \dots, k$ .

**Example 2.** The sum (24) has an interesting limit in case  $q_i \rightarrow 1$  for all  $i$ . In this case thanks to Remark 3 the terms where any of  $\bar{c}_j^i$  is nonvanishing vanishes. Then, denoting

$$\lim_{\substack{q_i \rightarrow 1 \\ i=1, \dots, k}} L_d^{\mathbb{E}} \left( b \mid \{n_i, q_i^{a_i}, q_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = L_{\mathbb{E}} \left( d \mid b \mid \{n_i, a_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right),$$

we obtain  $2k$ -parametric family

$$\mathbb{L}_d^{\mathbb{E}} \left( b \mid \{n_i, a_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = \sum_{\mathbf{c}^1, \dots, \mathbf{c}^k} H_{\Omega} \left( d; \mathbf{c}^1, \dots, \mathbf{c}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) \prod_{i=1}^k \frac{(-1)^{|\mathbf{c}^i|}}{a_i^{d|\mathbf{c}^i|}} (-n_i)_{|\mathbf{c}^i|} \prod_{j=1}^{d^*} \frac{a_i^{\ell(\Delta_j^i) c_j^i}}{c_j^i!} \quad (28)$$

Let us note that the exponent  $\sum_{j=1}^{d^*} \ell(\Delta_j^i) c_j^i$  has the meaning of the sum of lengths of all partitions of the color group  $i$  excluding the partition  $1^d$ , while  $|\mathbf{c}^i| := \sum_{j=1}^{d^*}$  has the meaning of total number of these partitions in this color group.

If we put  $\mathbb{E} = \mathbb{F} = 2$  and  $n_i = \pm 1$  in (28) we obtain the case considered in [45].

**Example 3.** We can further reduce (28) putting  $n_i = 1$  for  $i = 1, \dots, k$ . In this case each  $(-n_i)_{|\mathbf{c}^i|} = 0$  until  $|\mathbf{c}^i| = 1$ . It means that a single partition,  $\Delta_{j_i}^i$ , in each color group  $i$ , contributes to the set of the arguments of  $H_{d,N}^{\mathbb{E}}$ , and the summation in (28) is the summation over the set of various  $j_1, \dots, j_k$  where  $1 \leq j_i \leq d^*$ . Instead of (28) we obtain  $k$ -parametric family

$$\mathbb{L}_d^{\mathbb{E}} \left( b \mid \{1, a_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = \sum_{1 \leq j_1, \dots, j_k \leq d^*} H_{\Omega} \left( d; \Delta_{j_1}^1, \dots, \Delta_{j_k}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) \prod_{i=1}^k a_i^{\ell(\Delta_{j_i}^i) - d} \quad (29)$$

Relations (28)-(29) generates sums of Hurwitz numbers with certain conditions upon the partitions lengths. For instance, from (29) we obtain

$$\text{res}_{a_1=0} a_i^{d-l_1-1} \dots \text{res}_{a_k=0} a_k^{d-l_k-1} \mathbb{L}_d^{\mathbb{E}} \left( b \mid \{1, a_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = \sum H_{\Omega}(d, \Gamma_1, \dots, \Gamma_b, \Delta^1, \dots, \Delta^k, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})}) \quad (30)$$

where the sum is taken over all partitions  $\Delta^i$  of the fixed lengths  $l_i$ .

**Example 4.** One can re-write (28) as different  $2k$ -parametric family as follows. Using

$$(k)_{|\mathbf{c}^i|} = \frac{\Gamma(|\mathbf{c}^i| + k)}{\Gamma(k)}, \quad (1 - x_i)^{-|\mathbf{c}^i|} = 1 + \sum_{k=1}^{\infty} \frac{x_i^k}{k!} \frac{\Gamma(|\mathbf{c}^i| + k)}{\Gamma(|\mathbf{c}^i|)} = 1 + \sum_{k=1}^{\infty} \frac{x_i^k}{k!} \frac{(k)_{|\mathbf{c}^i|} \Gamma(k)}{\Gamma(|\mathbf{c}^i|)}$$

and taking the sum of (28) over  $n_i = -1, -2, \dots$  with the weight  $-\frac{x_i^{-n_i}}{n_i}$ ,  $x_i = 1 - z_i$ , we obtain

$$(-1)^k \sum_{n_1, \dots, n_k < 0} \prod_{i=1}^k \frac{(1 - z_i)^{-n_i}}{n_i} \mathbb{L}_d^{\mathbb{E}} \left( b \mid \{n_i, a_i\} \mid \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) = \sum_{\mathbf{c}^1, \dots, \mathbf{c}^k} H_{\Omega} \left( d; \mathbf{c}^1, \dots, \mathbf{c}^k, \Gamma_1, \dots, \Gamma_b, \Delta^{(1)}, \dots, \Delta^{(\mathbb{F})} \right) \prod_{i=1}^k \left( z_i^{-|\mathbf{c}^i|} - 1 \right) (-1)^{|\mathbf{c}^i|} \Gamma(|\mathbf{c}^i|) \prod_{j=1}^{d^*} \frac{a_i^{\ell(\Delta_j^i) c_j^i}}{c_j^i!} \quad (31)$$

where  $z_i$  and  $a_i$ ,  $i = 1, \dots, k$  are free parameters. Picking up terms at given powers of  $z_i$  and  $a_i$  one can obtain the sums of Hurwitz numbers under the conditions: the number of profiles in each color group,  $|\mathbf{c}^i|$ , is fixed and the sum of profile lengths inside each color group,  $\sum_{j=1}^{d^*} \ell(\Delta_j^i) c_j^i$ , is also fixed, see (67), (68) below. Let us note, that the term related to  $|\mathbf{c}^i| = 0$  should be treated as  $\lim_{|\mathbf{c}^i| \rightarrow 0} \left( z_i^{-|\mathbf{c}^i|} - 1 \right) \Gamma(|\mathbf{c}^i|) = -\log z$ .

### 3 Toda lattice and BKP tau functions

#### 3.1 Pochhammer symbols and content products

For a given partition  $\lambda = (\lambda_1, \dots, \lambda_l)$  and a function on the one-dimensional lattice  $r(x)$ ,  $x \in \mathbb{Z}$ , we introduce the generalized Pochhammer symbol  $r_n(x)$  as

$$r_n(x) = r(x)r(x+1) \cdots r(x+n-1) \quad (32)$$

and the generalized Pochhammer symbol,  $r_\lambda$ , related to a partition  $\lambda$  as

$$r_\lambda(x) = r_{\lambda_1}(x)r_{\lambda_2}(x-1) \cdots r_{\lambda_l}(x-l+1) \quad (33)$$

It may be written also as a content product as follows

$$r_\lambda(x) = \prod_{i,j \in \lambda} r(x+j-i)$$

where  $j-i$  is a content of the node in  $i$ -th row and  $j$ -th column of the Young diagram of a partition  $\lambda$ , see [29] for more details.

The content product plays an important role in many combinatorial problems, see for instance [15] where links to integrable systems were also pointed out.

**Remark 6.** (1) If  $r = fg$ , then  $r_\lambda(x) = f_\lambda(x)g_\lambda(x)$ . (2) If  $\tilde{r}(x) = (r(x))^n$ ,  $n \in \mathbb{C}$ , then  $\tilde{r}_\lambda(x) = (r_\lambda(x))^n$ .

**Example I.** For  $r(x) = x$  the generalized Pochhammer symbol coincides with the familiar one:

$$r_\lambda(x) = (x)_\lambda, \quad (x)_\lambda := (x)_{\lambda_1}(x-1)_{\lambda_1} \cdots (x-l+1)_{\lambda_l}, \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

If we take  $r(x) = x^n$ ,  $n \in \mathbb{C}$ , then  $r_\lambda(x) = ((x)_\lambda)^n$

**Example II.** For  $r(x) = 1 - qt^x$  the generalized Pochhammer symbol coincides with the  $q$ -deformed one (for some reasons we replace the letter  $q$  by the letter  $t$ ):

$$r_\lambda(x) = (qt^x; t)_\lambda, \quad (qt^x; t)_\lambda := (qt^x; t)_{\lambda_1}(qt^{x-1}; t)_{\lambda_1} \cdots (qt^{x-l+1}; t)_{\lambda_l}, \quad (qt^x; t)_n = (1-qt^x) \cdots (1-qt^{x+n-1})$$

This case may be also referred as trigonometric. The trigonometric  $r$  may be viewed as the infinite product of  $r$  from the Example I. If we take  $r(x) = (1 - qt^x)^n$ ,  $n \in \mathbb{C}$ , then  $r_\lambda(x) = ((qt^x; t)_\lambda)^n$ , see Remark 6.

**Example III.** For  $r(x) = \theta(cx, t)$ , where  $\theta(cx, t)$  is the Jacoby theta function,

$$\theta(cx, \tau) := \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n x) = (q; q)_\infty \prod_{n=1}^{\infty} \left(1 + q^{n-\frac{1}{2}} t^x\right) \left(1 + q^{n-\frac{1}{2}} t^{-x}\right)$$

where  $q = e^{2\pi i \tau}$ ,  $t = e^{2\pi i x}$ . As one can note the elliptic  $r$  is the infinite product of “trigonometric”  $r$  from the Example II. The generalized Pochhammer symbol is then the elliptic Pochhammer symbol:

$$r_\lambda(x) = [xc; t]_\lambda, \quad [cx; t]_\lambda := [cx; t]_{\lambda_1}[c(x-1); t]_{\lambda_1} \cdots [c(x-l+1); t]_{\lambda_l}, \quad [cx; t]_n := \theta(cx|t) \cdots \theta(c(x+n-1)|t)$$

If we take  $r(x) = (\theta(cx, t))^n$ ,  $n \in \mathbb{C}$ , then  $r_\lambda(x) = ([xc; t]_\lambda)^n$

The shall refer the cases  $r(x) = a + cx$ ,  $r(x) = 1 - qt^x$ ,  $r(x) = \theta(cx, \tau)$  as respectively rational, trigonometric and elliptic ones.

**Example IV.** For  $r(x) = e^{\beta x}$  we obtain the case considered in [1]

$$r_\lambda(x) = e^{\beta f_2(\lambda, x)}$$

where

$$f_2(\lambda, x) = \frac{1}{2} \sum_i \left[ \left(x + \lambda_i - i + \frac{1}{2}\right)^2 - \left(x - i + \frac{1}{2}\right)^2 \right], \quad f_2(\lambda, 0) + x|\lambda|$$

In [1] it was shown that  $f_2$  may be written in form (3) and this is a key relation to link the generating function for Hurwitz numbers to integrable systems.

**Examples of the parametrizations of  $r$**  It may be suitable to choose the function  $r$  as the product of functions  $r^{(1)} \dots r^{(k)}$  (then  $r_\lambda = r_\lambda^{(1)} \dots r_\lambda^{(k)}$ , see Remark 6), where each one is parametrized by the set of parameters  $\beta^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, \dots)$  in one of three ways:

$$(I) : \quad r^{(i)}(x) = e^{\beta_0^{(i)} + \sum_{m>0} \frac{1}{m} \beta_m^{(i)} x^m} \quad (34)$$

$$(II) : \quad r^{(i)}(x) = e^{\sum_{m>0} \frac{1}{m} \beta_m^{(i)} t_i^{mx}} \quad (35)$$

$$(III) : \quad r^{(i)}(x) = e^{\sum_{m>0} \frac{1}{m} \beta_m^{(i)} (e^{mc_i(a_i+x)} + e^{-mc_i(a_i+x)})} \quad (36)$$

To obtain respectively the usual, the trigonometric and elliptic Pochhammer symbols we choose

$$\beta_m^{(i)} = n_i (-a_i)^{-m}, \quad \beta_0^{(i)} = n_i \log a_i \quad (37)$$

$$\beta_m^{(i)} = -n_i q_i^m \quad (38)$$

$$\beta_m^{(i)} = n_i (-1)^m \frac{q_i^m}{1 - q_i^m} \quad (39)$$

Then respectively we obtain

$$r_\lambda(x) = \prod_{i=1}^k ((a_i + n)_\lambda)^{n_i} \quad (40)$$

$$r_\lambda(x) = \prod_{i=1}^k ((q_i t_i^{n_i}; t_i)_\lambda)^{n_i} \quad (41)$$

$$r_\lambda(x) = \prod_{i=1}^k ([c_i(a_i + x), q_i]_\lambda)^{n_i} \quad (42)$$

where  $a_i, q_i, t_i, n_i$  are complex numbers.

Now let us consider limiting expressions of rational, trigonometric and elliptic cases

$$r^{(i)}(x) = \left(1 + \frac{x}{n_i a_i}\right)^{n_i} \rightarrow \exp \frac{x}{a_i} \quad (43)$$

$$r^{(i)}(x) = \left(1 + \frac{q_i t_i^x}{n_i}\right)^{n_i} \rightarrow \exp q_i t_i^x \quad (44)$$

$$r^{(i)}(x) = (\theta(a_i, \tau_i))^{n_i} \rightarrow \exp q_i^{-\frac{1}{2}} (t_i^x + t_i^{-x}) \quad (45)$$

Formula similar to (43) was previously obtained in [47]. It result in the choice of  $r$  as in Example IV above (where we put  $\beta = \frac{1}{a}$ ), and allows to view the Okounkov tau function [1] as a limit  $n \rightarrow \infty$  of the certain hypergeometric function of matrix arguments [27] of type  $_{n+1}F_0$ .

Let us note that the variables  $\beta$  in the parametrization (36) is a triangle transform of variables  $t^*$  introduced in [32]. This follows from the relation  $r(x) = e^{U_{x-1} - U_x}$ , where  $U_x = \sum_{m \neq 0} t_m^* t^{mx}$ .

It actually means that  $\tau_r^{2KP}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)})$  is a 2KP tau function in variables  $(n, \mathbf{p}^{(1)}, \beta)$  in case  $p_m^{(2)} = \frac{1}{1-t^m}$ , see details in [46].

At last we need the important

**Lemma 1.**

$$s_\lambda(\mathbf{p}(q, t)) = (q; t)_\lambda s_\lambda(\mathbf{p}(0, t)) \quad (46)$$

and its limiting case

$$s_\lambda(\mathbf{p}(a)) = (a)_\lambda s_\lambda(\mathbf{p}_\infty) \quad (47)$$

where  $\mathbf{p}(q, t)$  is defined by

$$\mathbf{p}(q, t) = (p_1(q, t), p_2(q, t), \dots), \quad p_m(q, t) := \frac{1 - q^m}{1 - t^m} \quad (48)$$

$\mathbf{p}_\infty = (1, 0, 0, \dots)$  and

$$\mathbf{p}(a) = (a, a, \dots) \quad (49)$$

which may be easily obtained from the consideration in Ch I of [30]. This Lemma allows to identify series (5) where  $E = F = 1, 2$  with BKP and 2KP tau functions [26], [31].

### 3.2 TL tau function and TL hypergeometric tau function.

Here we recall few facts about the Toda lattice tau functions, and for details we refer to the paper [33]. The simplest Hirota equation for the Toda lattice is

$$\frac{\partial^2 \tau^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}})}{\partial p_1 \partial \bar{p}_1} \tau^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}}) - \frac{\partial \tau^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}})}{\partial p_1} \frac{\partial \tau^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}})}{\partial \bar{p}_1} = -\tau^{\text{TL}}(n+1, \mathbf{p}, \bar{\mathbf{p}}) \tau^{\text{TL}}(n-1, \mathbf{p}, \bar{\mathbf{p}}) \quad (50)$$

TL tau function may be written in form

$$\tau^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}}) = \sum_{\lambda \in \mathbf{P}} s_{\lambda}(\mathbf{p}) g_{\lambda, \mu}(n) s_{\mu}(\bar{\mathbf{p}}) \quad (51)$$

where  $g_{\lambda, \mu}$  is a determinant, see [34]. The Schur function is defined as follows

$$s_{\lambda}(\mathbf{p}) = \det(s_{\lambda_i - i + j}(\mathbf{p}))_{i, j}, \quad e^{\sum_{m>0} \frac{1}{m} z^m p_m} =: \sum_{m \geq 0} z^m s_m(\mathbf{p}) \quad (52)$$

We shall denote the length of a partition  $\lambda$  by  $\ell(\lambda)$  and the weight of  $\lambda$  by  $|\lambda|$ , see [30].

**TL tau function of the hypergeometric type.** These are

$$g(n) \sum_{\lambda \in \mathbf{P}} r_{\lambda}(n) s_{\lambda}(\mathbf{p}) s_{\lambda}(\bar{\mathbf{p}}) =: \tau_r^{\text{TL}}(n, \mathbf{p}) \quad (53)$$

In case  $r$  vanishes at certain site number  $M$  it is better to speak about

**Semi-infinite TL tau function** with the origin at the site number  $M$ . These are

$$g(n) \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq n-M}} r_{\lambda}(n) s_{\lambda}(\mathbf{p}) s_{\lambda}(\bar{\mathbf{p}}) =: \tau_r^{\text{TL}}(M, n, \mathbf{p}) \quad (54)$$

This tau function is a particular case of the previous one if we choose  $r(M) = 0$ .

In [26] there were written down two main examples of such tau functions:

$$\tau_r(n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \sum_{\lambda} (s_{\lambda}(\mathbf{p}_{\infty}))^{q-p} \frac{\prod_{i=1}^p s_{\lambda}(\mathbf{p}(a_i + n))}{\prod_{i=1}^q s_{\lambda}(\mathbf{p}(b_i + n))} s_{\lambda}(\mathbf{p}^{(1)}) s_{\lambda}(\mathbf{p}^{(2)}) \quad (55)$$

which may be related to the hypergeometric function of matrix argument [27], and

$$\tau_r(n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}) = \sum_{\lambda} (s_{\lambda}(\mathbf{p}_{\infty}))^{q-p} \frac{\prod_{i=1}^p s_{\lambda}(\mathbf{p}(t^{a_i+n}, t))}{\prod_{i=1}^q s_{\lambda}(\mathbf{p}(t^{b_i+n}, t))} s_{\lambda}(\mathbf{p}^{(1)}) s_{\lambda}(\mathbf{p}^{(2)}) \quad (56)$$

which which may be related to the Milne's hypergeometric functions [28].

### 3.3 BKP tau function.

We are interested in a certain subclass of the BKP tau functions (59) written down in [31] and called BKP hypergeometric tau functions, and also in the similar class of TL tau functions (51) found in [25], [26].

There are two different BKP hierarchies of integrable equations, one was introduced by the Kyoto group in [23], the other was introduced by V. Kac and J. van de Leur in [21]. We need the last one. This hierarchy includes the celebrated KP one as a particular reduction. In a certain way (see [42]) the BKP hierarchy may be related to the three-component KP hierarchy introduced in [23] (earlier described in [24] with the help of L-A pairs with matrix valued coefficients). For a detailed description of the BKP we refer readers to the original work [21], and here we write down the first non-trivial equations for the BKP tau function (Hirota equations). These are

$$\begin{aligned} & \frac{1}{2} \frac{\partial \tau(N, n, \mathbf{p})}{\partial p_2} \tau(N+1, n, \mathbf{p}) - \frac{1}{2} \tau(N, n, \mathbf{p}) \frac{\partial \tau(N+1, n, \mathbf{p})}{\partial p_2} + \frac{1}{2} \frac{\partial^2 \tau(N, n, \mathbf{p})}{\partial^2 p_1} \tau(N+1, n, \mathbf{p}) \\ & + \frac{1}{2} \tau(N, n, \mathbf{p}) \frac{\partial^2 \tau(N+1, n, \mathbf{p})}{\partial^2 p_1} - \frac{\partial \tau(N, n, \mathbf{p})}{\partial p_1} \frac{\partial \tau(N+1, n, \mathbf{p})}{\partial p_1} = \tau(N+2, n, \mathbf{p}) \tau(N-1, n, \mathbf{p}) \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{1}{2} \tau(N, n+1, \mathbf{p}) \frac{\partial^2 \tau(N+1, n, \mathbf{p})}{\partial^2 p_1} - \frac{1}{2} \frac{\tau(N, n+1, \mathbf{p})}{\partial^2 p_1} \tau(N+1, n, \mathbf{p}) = \\ \frac{\partial \tau(N+2, n, \mathbf{p})}{\partial p_1} \tau(N-1, n+1, \mathbf{p}) - \frac{\partial \tau(N+1, n+1, \mathbf{p})}{\partial p_1} \tau(N, n, \mathbf{p}) \end{aligned} \quad (58)$$

The BKP tau functions depend on the set of higher times  $t_m = \frac{1}{m} p_m$ ,  $m > 1$  and the discrete parameter  $N$ . In [31] the second discrete parameter  $n$  was added and equation (58) relates BKP tau functions with neighboring  $n$ . The complete set of the Hirota equations with two discrete parameters is written down in the Appendix.

The general solution to Hirota equations may be written as

$$\tau^{BKP}(N, n, \mathbf{p}) = \sum_{\lambda \in P} A_\lambda(N, n) s_\lambda(\mathbf{p}) \quad (59)$$

where  $P$  is the set of all partitions and where  $A_\lambda$  solves Plucker relations for isotropic Grassmannian and may be written in a pfaffian form.

**BKP tau function of the hypergeometric type** We consider sums over partitions of form

$$g(n) \sum_{\substack{\lambda \in P \\ \ell(\lambda) \leq N}} r_\lambda(n) s_\lambda(\mathbf{p}) =: \tau_r^{BKP}(N, n, \mathbf{p}) \quad (60)$$

where  $P$  is the set of all partitions,  $s_\lambda$  are the Schur functions [30] and the semi-infinite set  $\mathbf{p} = (p_1, p_2, \dots)$  is related to the called higher times in the soliton theory  $\mathbf{t} = (t_1, t_2, \dots)$  via  $p_m = m t_m$ . The constant  $g(n)$  is not important and may be found in Appendix A.3, see (99),(100).

**Remark 7.** For hypergeometric tau functions (53) and (60) there is an obvious transformation  $r_\lambda \rightarrow a^{-|\lambda|} r_\lambda$ ,  $p_m^1 \rightarrow a p_m^{(1)}$ ,  $m > 0$ , which does not change the tau functions.

**Remarks.** According to [26], [31]

(1)  $\tau_r^{TL}$  solves certain linear equation generalizing Gauss equation for Gauss hypergeometric function which may be also referred as a “string equation”

(2) There are various determinantal formulae to present  $\tau_r^{TL}$ , and pfaffian formulae to present  $\tau_r^{BKP}$

(3) Both  $\tau_r^{TL}$  and  $\tau_r^{BKP}$  may be obtained by the action of vertex operators on certain simple functions

Also [31, 35, 36] :

(4) Sums (60) and (53) may be considered as partition functions for models of random partitions where a partition  $\lambda$  contributes the weights  $r_\lambda(n) s_\lambda(\mathbf{p})$ , or  $r_\lambda(n) s_\lambda(\mathbf{p}) s_\lambda(\bar{\mathbf{p}})$  respectively

(5) For certain specifications of  $r$  and  $\mathbf{p}$  sums (60) and (53) may be viewed as multisoliton tau functions. Similarly, for the same specifications, they may be viewed as discrete versions of matrix models

## 4 Generating Hurwitz series and the BKP and 2KP tau functions

As we have seen

$$\begin{aligned} \tau^{E,F} \left( N, n, \beta, \{n_i, q_i, t_i\} \mid \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F)} \right) = \\ \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_\lambda^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) \prod_{i=1}^F s_\lambda(\mathbf{p}^{(i)}) = \end{aligned}$$

$$\sum_{d \geq 0} \sum_{\substack{\Delta^{(1)}, \dots, \Delta^{(F)} \in \mathbf{P} \\ |\Delta^{(j)}| = d, j=1, \dots, F}} \frac{\beta^b}{b!} L_{d,N}^{\mathbf{E}, \mathbf{F}} \left( b \mid \{n_i, q_i t_i^n, t_i\} \mid \Delta^{(1)}, \dots, \Delta^{(F)} \right) \prod_{j=1}^F \mathbf{p}_{\Delta^{(j)}}^j$$

where

$$r_{\lambda}^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) = (qe^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} \prod_{i=1}^k \left( \frac{(q_i t_i^n; t_i)_{\lambda}}{(1 - q_i)^d} \right)^{n_i}$$

and where  $L_{d,N}^{\mathbf{E}, \mathbf{F}}(b \mid \{n_i, q_i t_i^n, t_i\} \mid \Delta^{(1)}, \dots, \Delta^{(F)})$  for  $N \geq d$  is the weighted sum of Hurwitz numbers for the  $d$ -fold covering of the surface of Euler characteristic  $\mathbf{E}$  with fixed ramification profiles  $\Delta^{(1)}, \dots, \Delta^{(F)}$  and  $b$  profiles  $\Gamma = 1^{d-2}2$ , and summation runs over additional profiles as it was described in Section 2.2.

Let  $\mathbf{E} = \mathbf{F}$ . We have

**Theorem 1.** *For any complex numbers  $\{n_i, q_i, t_i\}$  and integers  $n$  and  $N \geq 0$  the generating Hurwitz series*

$$\begin{aligned} & \tau^{1,1}(N, n, \beta, \{n_i, q_i^{a_i}, q_i\} \mid \mathbf{p}) \\ &= \sum_{d \geq 0} \sum_{\substack{\Delta \in \mathbf{P} \\ |\Delta| = d}} \frac{\beta^b}{b!} L_{d,N}^{1,1}(b \mid \{n_i, q_i t_i^n, t_i\} \mid \Delta) \mathbf{p}_{\Delta} \\ &= \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_{\lambda}^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) s_{\lambda}(\mathbf{p}) \end{aligned}$$

where

$$r_{\lambda}^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) = (qe^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} \prod_{i=1}^k \left( \frac{(q_i t_i^n; t_i)_{\lambda}}{(1 - q_i)^d} \right)^{n_i} \quad (61)$$

is a  $\tau$ -function of the BKP hierarchy where  $\mathbf{p} = (p_1, p_2, \dots)$  plays the role of higher times.

**Theorem 2.** *For any complex numbers  $\{n_i, q_i, t_i\}$  and integers  $n$  and  $N \geq 0$  the generating Hurwitz series*

$$\begin{aligned} & \tau^{2,2}(N, n, \beta, \{n_i, q_i^{a_i}, q_i\} \mid \mathbf{p}^{(1)}, \mathbf{p}^{(2)}) \\ &= \sum_{d \geq 0} \sum_{\substack{\Delta^{(1)}, \Delta^{(2)} \in \mathbf{P} \\ |\Delta^{(j)}| = d, j=1, 2}} \frac{\beta^b}{b!} L_{d,N}^{2,2}(b \mid \{n_i, q_i t_i^n, t_i\} \mid \Delta^{(1)}, \Delta^{(2)}) \prod_{j=1,2} \mathbf{p}_{\Delta^{(j)}}^j \\ &= \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_{\lambda}^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) \prod_{i=1,2} s_{\lambda}(\mathbf{p}^{(i)}) \end{aligned}$$

where

$$r_{\lambda}^{\mathbf{n}, \mathbf{q}, \mathbf{t}}(n) = (qe^{\beta n})^{|\lambda|} e^{\beta f_2(\lambda)} \prod_{i=1}^k \left( \frac{(q_i t_i^n; t_i)_{\lambda}}{(1 - q_i)^d} \right)^{n_i} \quad (62)$$

is a  $\tau$ -function of the 2-component BKP hierarchy and also of the semiinfinite TL hierarchy where  $\mathbf{p}^{(j)} = (p_1^{(j)}, p_2^{(j)}, \dots)$  play the role of higher times.

*Proof.* . The statements of the Theorems directly follow from the consideration in the previous sections, see Proposition 1 and formulae (60) and (53) where we choose

$$r(x) = e^{\beta x} \prod_{i=1}^k (1 - q_i t_i^x)^{n_i} \quad (63)$$

This choice provides

$$r_{\lambda}(x) = e^{\beta f_2(x)} \prod_{i=1}^k ((q_i t_i^x; t_i)_{\lambda})^{n_i} \quad (64)$$

□

Let us mark that if we introduce

$$F(\mathbf{q}, \mathbf{t}, \mathbf{n}; \beta, x) = \sum_{m>0} e^{\beta m} \prod_{i=1}^k (1 - q_i t_i^m)^{n_i} x^m \quad (65)$$

then

$$\tau^{2,2}(N, n, \beta, \{n_i, q_i^{a_i}, q_i\} | X, Y) = \det \left[ F(\mathbf{q}, \mathbf{t}, \mathbf{n}; \beta, x_i^{(1)} x_j^{(2)}) \right]_{i,j=1,\dots,N}, \quad p_m^{(j)} = \sum_{i=1}^N \left( x_i^{(j)} \right)^m$$

#### 4.1 Particular cases

We consider the sums of Hurwitz numbers which correspond to the following options:

- $b, d, k \in \mathbb{Z}$ , where  $d > 0$ ,  $b, k \geq 0$ ;
- integers  $N_1, \dots, N_k \geq 0$ ;
- integers  $L_1, \dots, L_k \geq 0$ ;
- integers  $c_j^i \geq 0$ , where  $i = 1, \dots, k$ ,  $j = 1, \dots, d^*$ ;
- partitions  $\Delta_j^i$ , where  $i = 1, \dots, k$ ,  $j = 1, \dots, d^*$  and  $|\Delta_j^i| = d$ ;
- partitions  $\Delta^{(1)}, \dots, \Delta^{(F)}$ , where  $|\Delta^{(i)}| = d$ ;

Denote by

$$T_{d,N}^E(b|N_1, \dots, N_k | L_1, \dots, L_k | \Delta^{(1)}, \dots, \Delta^{(F)}) = \sum \frac{1}{\prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq d^*}} c_j^i!} H_{d,N}^{E,F+b+\sum_i |c^i|}(\Gamma_1, \dots, \Gamma_b, c_1^1, \dots, c_{d^*}^1, \dots, c_1^k, \dots, c_{d^*}^k, \Delta^{(1)}, \dots, \Delta^{(F)}), \quad (66)$$

where the sum is taken over partitions  $\Delta_i$  of the weight  $d$  and over all partitions  $\Delta_j^i$  of the same weight that are not equal to  $(1, \dots, 1)$ , such that

- $\Gamma_1 = \dots = \dots, \Gamma_b = [2, 1, \dots, 1]$ ;
- $\sum_{j=1}^{d^*} c_j^i = N_i, \quad i=1, \dots, k$
- $\sum_{j=1}^{d^*} c_j^i \ell(\Delta_j^i) = L_i, \quad i=1, \dots, k$ .

We recall that for  $N \geq d$  the numbers  $H_{d,N}^{E,p}$  are the Hurwitz numbers of the  $d$ -fold coverings with  $p$  branch points of the surface with Euler characteristic  $E$ .

Using the Example 4 in Subsection 2.2 we obtain the following corollary of the Theorem 1 we get

**Corollary 1.**

$$\begin{aligned} & (-1)^k \sum_{n_1, \dots, n_k < 0} \prod_{i=1}^k \frac{(1 - z_i)^{-n_i}}{n_i} \tau^{1,1}(N, n, \mathbf{p}, \beta | \{n_i, a_i\}) = \\ & \sum_{\Delta} e^{\beta |\lambda| n} \mathbf{p}_{\Delta} \frac{\beta^b}{b!} \times \\ & T_{d,N}^{(1)}(b|N_1, \dots, N_k | L_1, \dots, L_k | \Delta^{(1)}, \dots, \Delta^{(2)}) \prod_{i=1}^k \left( z_i^{-N_i} - 1 \right) (-1)^{N_i} \Gamma(N_i) (a_i + n)^{L_i} \end{aligned} \quad (67)$$



where  $z_i$  and  $a_i$ ,  $i = 1, \dots, k$  are free parameters and where

$$\tau^{1,1}(N, n, \mathbf{p}, \beta | \{n_i, a_i\}) = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} e^{\beta f_2(\lambda, n)} \prod_{i=1}^k ((a_i + n)_\lambda)^{n_i} s_\lambda(\mathbf{p})$$

is the BKP tau function.

By the Theorem 2

**Corollary 2.**

$$\begin{aligned} (-1)^k \sum_{n_1, \dots, n_k < 0} \prod_{i=1}^k \frac{(1 - z_i)^{-n_i}}{n_i} \tau^{2,2}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)} | \{n_i, a_i\}) = \\ \sum_{\Delta^{(1)}, \Delta^{(2)}} e^{\beta |\lambda| n} \mathbf{p}_{\Delta^{(1)}}^{(1)} \mathbf{p}_{\Delta^{(2)}}^{(2)} \frac{\beta^b}{b!} \times \\ T_{d,N}^E(b | N_1, \dots, N_k | L_1, \dots, L_k | \Delta^{(1)}, \Delta^{(2)}) \prod_{i=1}^k \left( z_i^{-N_i} - 1 \right) (-1)^{N_i} \Gamma(N_i) (a_i + n)^{L_i} \end{aligned} \quad (68)$$

where  $z_i$  and  $a_i$ ,  $i = 1, \dots, k$  are free parameters and where

$$\tau^{2,2}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)} | \{n_i, a_i\}) = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} e^{\beta f_2(\lambda, n)} \prod_{i=1}^k ((a_i + n)_\lambda)^{n_i} s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)})$$

is the 2KP (or, the same, of the semiinfinite TL) tau function.

**Alternating sums.** Here we present other combination of Hurwitz numbers weighted by  $\pm 1$ .

We consider the sums of Hurwitz numbers which correspond to the following options:

- $b, d, k, s \in \mathbb{Z}$ , where  $d > 0$ ,  $b, k, s \geq 0$ ;
- $l_1, \dots, l_k \in \mathbb{Z}$ , where  $0 \leq l_i \leq d$ ;
- $l_1^*, \dots, l_s^* \in \mathbb{Z}$ , where  $0 < l_i^* \leq d$  (the first unicity is strict);
- partitions  $\Delta^{(1)}, \dots, \Delta^{(F)}$ , where  $|\Delta^{(i)}| = d$ ;
- partitions  $\Delta_1, \dots, \Delta_k$ , where  $|\Delta_i| = d$ ;
- $s$  sets of partitions  $\Delta_1^i, \dots, \Delta_{m_i}^i$ ,  $i = 1, \dots, s$ , where  $|\Delta_j^i| = d$  for each  $i, j$ .

Denote by

$$S_{d,N}^E(b | l_1, \dots, l_k | l_1^*, \dots, l_s^* | \Delta^{(1)}, \dots, \Delta^{(F)}) =$$

$$\sum (-1)^\varepsilon H_{d,N}^E(\Gamma_1, \dots, \Gamma_b, \Delta_1, \dots, \Delta_k, \Delta_1^1, \dots, \Delta_{m_1}^1, \dots, \Delta_1^s, \dots, \Delta_{m_s}^s, \Delta^{(1)}, \dots, \Delta^{(F)}), \quad (69)$$

where the sum is taken over partitions  $\Delta_i$  of the weight  $d$  and over all partitions  $\Delta_j^i$  of the same weight that are not equal to  $(1, \dots, 1)$ , such that

- $\Gamma_1 = \dots = \Gamma_b = [2, 1, \dots, 1]$ ;
- $d - \ell(\Delta_i) = l_i$ ,  $i = 1, \dots, k$ ;
- $\sum_{j=1}^{m_i} (d - \ell(\Delta_j^i)) = l_i^*$ ,  $i = 1, \dots, s$ ;
- $\varepsilon = m_1 + \dots + m_s$ .

Thus, the numbers  $S_{d,N}^E(b|1, \dots, 1|1, \dots, 1|\Delta^{(1)}, \Delta^{(2)})$  are the 2-Hurwitz numbers from [1] with profiles  $\Delta^{(1)}$  and  $\Delta^{(2)}$  in two given points with additional  $b + k + s$  simple ramification points.

It follows from (16) that

$$S_{d,N}^E(b|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(F)})$$

we can present as the sum, where summands correspond to Young diagram  $\lambda$  of weight  $d$ .

**Hurwitz weighted sums (69) for the BKP hierarchy.** Let us write down the following particular case of the Theorem 1. Construct now generating function for  $S_{\mathbb{RP}^2}^N(d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta)$ . Put

$$\tau^{BKP}(N, n|a_1, \dots, a_k|b_1, \dots, b_s|\mathbf{p}_\Delta) = \sum_B (qe^{\beta n})^d \frac{\beta^b}{b!} \prod_{i=1}^s b_i^{-d} (b_i + n)^{-l_i^*} \prod_{i=1}^k a_i^{-d} (a_i + n)^{-l_i} S_{\mathbb{RP}^2}^N(B) p_\Delta \quad (70)$$

where the sum is taken over all  $B = (d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta)$ . Here  $p_\Delta = p_{d_1} p_{d_2} \dots$  with  $\Delta = (d_1, d_2, \dots)$  and  $\Delta = (d_1, d_2, \dots)$ ,  $d_1 + d_2 + \dots = d$ .

**Corollary 3.** *The function  $\tau^{BKP}(N, n|a_1, \dots, a_k|b_1, \dots, b_s|\mathbf{p})$  is a  $\tau$ -functions for BKP hierarchy for any complex numbers  $(a_1, \dots, a_k|b_1, \dots, b_s)$  and integer  $n > 0$ .*

**Hurwitz weighted sums (69) for the 2KP (the same, for the semiinfinte TL) hierarchy.** Consider

$$\tau^{TL}(M, n|q, \beta|a_1, \dots, a_k|b_1, \dots, b_s|\mathbf{p}, \bar{\mathbf{p}}) = \quad (71)$$

$$\sum_B (qe^{\beta n})^d \frac{\beta^b}{b!} \prod_{i=1}^k a_i^{-d} (a_i + n)^{-l_i} \prod_{i=1}^s b_i^{-d} (b_i + n)^{-l_i^*} S_{\mathbb{CP}^1}^{n-M}(B|\Delta^{(1)}, \Delta^{(2)}) p_{\Delta^{(1)}} \bar{p}_{\Delta^{(2)}} \quad (72)$$

where  $n > M$  and the sum is taken over all  $B = (d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \Delta^{(2)})$ . Here  $p_{\Delta^{(i)}} = p_{d_1^{(i)}} p_{d_2^{(i)}} \dots$  with  $\Delta^{(i)} = (d_1^{(i)}, d_2^{(i)}, \dots)$ ,  $d_1^{(i)} + d_2^{(i)} + \dots = d$ ,  $i = 1, 2$ .

Here we basically review the result of [45] with certain modifications, namely, we need not infinite but semiinfinite Toda lattice to compare with our main result in the next section. We do not consider the combinatorial interpretation of hypergeometric tau functions in terms of counting paths problem in Cayley graph related to the symmetric group worked out in [22], [45].

**Corollary 4.** *The function  $\tau^{TL}(M, n|a_1, \dots, a_k|b_1, \dots, b_s|\mathbf{p}, \bar{\mathbf{p}})$  is a  $\tau$  functions for semiinfinte 2DToda hierarchy for any complex numbers  $(a_1, \dots, a_k|b_1, \dots, b_s)$ .*

It directly follows from the Theorem 2.

**Remark 8.** The special case  $s = 0$  was pointed out in [16].

## 5 Transformation of Hurwitz $\tau$ -functions for the semiinfinte 2DToda hierarchy to $\tau$ -functions for BKP hierarchy

The sum (60) is an example of the BKP hypergeometric tau function [31] (Hirota equations for the BKP tau functions may be found in Appendix A.2). It may be obtained from the hypergeometric tau function of the semi-infinite Toda lattice  $n \geq M$  (here a given  $M$  is the origin of the lattice):

$$\frac{\partial \phi_n}{\partial p_1 \partial \bar{p}_1} = r'(n) e^{\phi_{n-1} - \phi_n} - r'(n+1) e^{\phi_n - \phi_{n+1}}, \quad e^{-\phi_n} = \frac{\tau_{r'}(M; n+1, \mathbf{p}, \bar{\mathbf{p}})}{\tau_{r'}(M; n, \mathbf{p}, \bar{\mathbf{p}})} \frac{g(n)}{g(n+1)}$$

where  $r'(n) = r(n)\delta(n)$  ( $\delta(M) = 0$ ,  $\delta(n) = 1$  otherwise). The multiplication by  $\delta$  provides the restriction of the summation region by the condition  $r'_\lambda(n)$  for  $\ell(\lambda) \leq n - M$  for the tau function of the Toda lattice, this we see from the definition of  $r_\lambda(n)$ . Fixing  $n$  and choosing  $N = n - M$  we obtain

$$\tau_r^{BKP}(N, n, \mathbf{p}) = \left[ e^{\frac{1}{2}L_\infty} \cdot \tau_{r'}^{TL}(n - N; n, \mathbf{p}, \bar{\mathbf{p}}) \right]_{\bar{\mathbf{p}}=0} \quad (73)$$

where  $L_\infty$  is the following Laplacian operator

$$L_\infty = \sum_{m \geq 1} m \frac{\partial^2}{\partial \bar{p}_m^2} + 2 \sum_{m \geq 1, \text{odd}} m \frac{\partial}{\partial \bar{p}_m} \quad (74)$$

The operator  $e^{\frac{1}{2}L_\infty}$  and the evaluation at  $\bar{\mathbf{p}} = 0$  'eliminate' one Schur function in each term of (53). The proof of (73) follows from

$$\sum_{\lambda \in \mathbf{P}} s_\lambda(\mathbf{p}) = e^{\frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} p_m^2 + \sum_{m=1}^{\infty} p_{2m-1}}$$

and

$$\left[ s_\mu(\tilde{\partial}) \cdot s_\lambda(\mathbf{p}) \right]_{\mathbf{p}=0} = \delta_{\mu, \lambda}$$

which may be derived from Examples in Chapter I section 5 of [30]. Here  $s_\lambda(\tilde{\partial})$  denotes the Schur function as the function defined by (52) where each  $p_m$  is replaced by  $m \frac{\partial}{\partial p_m}$ .

## 6 Matrix integrals as generating functions of Hurwitz numbers and of Hurwitz generating series

We want to present Hurwitz generating series (5) in form of matrix integrals. The idea is that various integrals of products of the 2KP and BKP tau functions of matrix argument <sup>2</sup> result in the series (5).

**One-matrix model and Hurwitz numbers.** To begin with we recall that the partition function of such standard matrix models as the two-matrix model, the model of complex matrices (complex Ginibre ensemble) and model of normal matrices may be written in form of the following perturbation series in coupling constants  $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}$

$$\sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \frac{s_\lambda(I_N)}{s_\lambda(\mathbf{p}_\infty)} s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)}) \quad (75)$$

see [39], [32], [36]. Taking into account explanations above, and

$$s_\lambda(I_N) = \chi_\lambda(1) N^d \left( 1 + \sum_{\Delta \neq 1^d} |C_\Delta| \frac{\chi_\lambda(\Delta)}{\chi_\lambda(1)} N^{\ell(\Delta)-d} \right)$$

This series is of form (5) where  $k = 1$  and where  $q_1 = t_1^N \rightarrow 1$  which, as we know, generates sums of Hurwitz numbers for the sphere with three ramification points where two profiles are arbitrary and fixed and summation is ranged over all profiles in the third point whose length is a given number. At first this fact was marked in [16].

Our remark to this observation is as follows. Series (75) also describes the celebrated one-matrix model in case  $p_2^{(2)} \neq 0, p_i^{(2)} = 0, i > 2$ , for the Schur function seires see [37]. Thus we obtain

- The one-matrix model generates sums of all Hurwitz numbers for the sphere under the following conditions: the profile at  $\infty$  is a given partition, the profile at 0 is a given partition of type  $1^{d_1} 2^{d_2}$ , the length of the profile at the third point is a given number.

**Remark 9.** To compare results, one should have in mind that in matrix model studies they typically send  $p_i \rightarrow N p_i, \bar{p}_i \rightarrow N \bar{p}_i$  in the  $N \rightarrow \infty$  limit which yields the following behavior in  $N$ :

$$s_\lambda(\mathbf{p}) = \chi_\lambda(1) N^d \left( p_1^d + \sum_{\Delta \neq 1^d} |C_\Delta| \frac{\chi_\lambda(\Delta)}{\chi_\lambda(1)} N^{\ell(\Delta)-d} \mathbf{p}_\Delta \right)$$

---

<sup>2</sup> If  $\mathbf{p} = (p_1, p_2, \dots)$  and  $p_m = \text{tr} M^m$  where  $M$  is a matrix, we may write  $\tau = \tau(M) := \tau(\mathbf{p}(M))$  and call it tau function of matrix argument

**Example.** The partition function for the Hermitian one-matrix model which is related to the 'triangulation' by  $k$ -gons and describes a model of two-dimensional gravity is

$$\begin{aligned} Z(N, g, g_k) &= \int dM e^{-N \text{tr}(\frac{g}{2} M^2 + \frac{g_k}{k} M^k)} = \frac{s_\lambda(I_N)}{s_\lambda(\mathbf{p}_\infty)} s_\lambda \left( 0, p_2 = -\frac{1}{2Ng}, 0, \dots \right) s_\lambda(0, \dots, 0, p_k = -Ng_k, 0, \dots) \\ &= \sum_{F, L, V} Z_{F, L, V} N^{F-L+V} g^{-L} g_k^V \end{aligned} \quad (76)$$

where  $F, L, V$  are the number of faces, lines and vertices of the Feynman fat graph which contributes to  $Z_{F, L, V}$ .<sup>3</sup> (We choose the normalization of the matrix integral in such a way that  $Z(N, g, g_k)$  is equal to 1 when  $g_k = 0$ .) Thus we obtain

$$Z_{F; L; V} = \sum_{\substack{\Delta \\ \ell(\Delta) = F}} H_{\mathbb{CP}^1}(\Delta, (2^L), (k^V)) \quad (77)$$

where  $|\Delta| = 2L = kV$ , otherwise both sides vanish. After taking the logarithm we obtain the same for the connected Feynman graphs and the connected Hurwitz numbers:

$$Z_{F; L; V}^{\text{connected}} = \sum_{\substack{\Delta \\ \ell(\Delta) = F}} H_{\mathbb{CP}^1}^{\text{connected}}(\Delta, (2^L), (k^V)) \quad (78)$$

In case there is a set of coupling constants  $g_3, g_4, \dots$  instead of a single one,  $g_k$ , we obtain

$$Z(N, g, g_3, \dots) = \sum_{F, L, V} Z_{F, L, V} N^{F-L+\sum_{k \geq 2} V_k} g^{-L} \prod_{k \geq 2} g_k^{V_k}$$

where  $Z_{F; L; V_3, V_4, V_5, \dots}$  is equal to the sum of the 3-point Hurwitz numbers,

$$Z_{F; L; V_3, V_4, V_5, \dots} = \sum_{\substack{\Delta \\ \ell(\Delta) = F}} H_{\mathbb{CP}^1}(\Delta, \Delta_1, \Delta_2), \quad \Delta_1 = (2^L), \Delta_2 = (3^{V_3} 4^{V_4} 5^{V_5} \dots) \quad (79)$$

and the Euler characteristic of related fat graphs is equal to the alternating sum of the lengths of profiles:  $E_{\text{fat graph}} = \ell(\Delta) - \ell(\Delta_1) + \ell(\Delta_2)$ . We also obtain  $|\Delta| = |\Delta_1| = |\Delta_2|$ .

In [16] some other interesting examples of relations between matrix models and sums of Hurwitz numbers were mentioned. For example it was presented a  $2n$ -matrix model with expansion (75) where  $\frac{s_\lambda(I_N)}{s_\lambda(\mathbf{p}_\infty)}$  was replaced by the  $n$ -th power of this factor.

Below we will write down matrix integrals which generate Hurwitz numbers themselves rather than weighted sums (as (75) does), though we also present some generating series for the sums.

**Notation. Useful relations.** We shall exploit the known formulae for integrations of Schur functions over the unitary group and over complex matrices. Earlier they were used in [32] to clarify some links between matrix models and integrable hierarchies.

As in [32] instead of power sums written by small bold characters (like  $\mathbf{p}$ ) sometimes where it is suitable we shall use the matrix arguments written by large character. Say,  $s_\lambda(X)$  and  $\tau(X)$  are respectively equal to  $s_\lambda(X) := s_\lambda(\mathbf{p}(X))$  and  $\tau(X) := \tau(\mathbf{p}(X))$  where  $\mathbf{p}(X) = (p_1(X), p_2(X), \dots)$ , where  $p_m(X) = \text{tr} X^m$ .

We use the following notations

- $d_* U$  is the normalized Haar measure on  $\mathbb{U}(N)$ :  $\int_{\mathbb{U}(N)} d_* U = 1$

---

<sup>3</sup> According to the Feynman rules for the one matrix model we have: (a) to each propagator (double line) is associated a factor  $1/(Ng)$  (which is  $p_2^{(1)}$  in our notations) (b) to each four legged vertex is associated a factor  $(-Ng_4)$  (which is  $Np_4^{(2)}$  in our notations) (c) to each closed single line is associated a factor  $N$ . Therefore we may say that the factor  $\frac{s_\lambda(I_N)}{s_\lambda(\mathbf{p}_\infty)}$  in (75) is responsible for closed lines, the factor  $s_\lambda(\mathbf{t} = 0, Np_2^{(1)}, 0, \dots)$  is responsible for propagators and the factor  $s_\lambda(N\mathbf{p}^{(2)})$  is responsible for vertices.

- $Z$  is a complex matrix,  $Z = UX(1+J)U^\dagger$  (the Schur decomposition), where  $X = \text{diag}(z_i)$  is diagonal,  $J$  is strictly upper triangle,  $U \in \mathbb{U}(N)$

$$\begin{aligned} d\Omega^{\mathbb{C}}(Z, Z^\dagger) &= \pi^{-n^2} e^{-\text{tr}(ZZ^\dagger)} \prod_{i,j=1}^N d\Re Z_{ij} d\Im Z_{ij} \\ &= c_Z d_* U \prod_{N \geq i > j} |z_i - z_j|^2 \prod_{i=1}^N e^{-|z_i|^2} d^2 z_i \left[ e^{-\text{tr} J J^\dagger} d^2 J_{ij} \right] \end{aligned}$$

where the part related to the upper triangular factor in brackets is not important for our problems.

- $M$  is a normal matrix,  $Z = UXU^\dagger$ , where  $X = \text{diag}(z_i)$  is diagonal,  $U \in \mathbb{U}(N)$

$$\begin{aligned} d\Omega^{\mathbb{N}}(M, M^\dagger) &= \pi^{-n^2} e^{-\text{tr}(MM^\dagger)} \prod_{i,j=1}^N d\Re M_{ij} d\Im M_{ij} \\ &= c_M d_* U \prod_{N \geq i > j} |z_i - z_j|^2 \prod_{i=1}^N e^{-|z_i|^2} d^2 z_i \end{aligned}$$

- $H^{(1)}$  is a Hermitian matrix and  $H^{(2)}$  is anti-Hermitian one,  $H^{(c)} = U^{(c)} X^{(c)} U^{(c)\dagger}$ ,  $X^{(c)} = \text{diag}(x_i^{(c)})$ ,  $U, U^{(c)} \in \mathbb{U}(N)$ ,  $c = 1, 2$ . Measure

$$\begin{aligned} d\Omega^{\mathbb{H}}(H^{(1)}, H^{(2)}) &= \int_{\mathbb{U}(N)} e^{-\text{tr}(H^{(1)} U H^{(2)} U^\dagger)} d_* U \prod_{i \leq j} d\Re H^{(1)} d\Im H^{(2)} \prod_{i < j} d\Im H^{(1)} d\Re H^{(2)} \\ &= c_H \prod_{c=1,2} d_* U^{(c)} \prod_{N \geq i > j} (x_i^{(c)} - x_j^{(c)}) \prod_{i=1}^N e^{-x_i^{(1)} x_i^{(2)}} dx_i^{(1)} dx_i^{(2)} \end{aligned}$$

where the constants  $c_a$ ,  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ , are chosen for normalization:  $\int d\Omega_p^{(a)} = 1$ .

**Remark 10.** In what follows, for unification and to save space, we shall use the notation  $M$  and  $M^*$  replacing the pairs  $Z, Z^\dagger$ ,  $M, M^\dagger$  and also  $H^{(1)}, H^{(2)}$ . In the last case the matrices  $M$  and  $M^*$  are not related by the Hermitian conjugation.

These measures provides the relation

$$\int s_\lambda(M) s_\mu(M^*) d\Omega^a(M, M^*) = (N)_\lambda \delta_{\lambda, \mu} \quad (80)$$

where  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ . This relation was used in [38], [39], [32], [16], [36], for models of Hermitian, complex and normal matrices.<sup>4</sup>

By  $I_N$  we shall denote the  $N \times N$  unit matrix. Then (for instance see [30])

**Lemma 2.** *Let  $A$  and  $B$  be normal matrices (i.e. matrices diagonalizable by unitary transformations). Then*

$$\int_{\mathbb{U}(N)} s_\lambda(AUBU^{-1}) d_* U = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(I_N)}, \quad (81)$$

For  $A, B \in GL(N)$  we have

$$\int_{\mathbb{U}(n)} s_\mu(AU) s_\lambda(U^{-1}B) d_* U = \frac{s_\lambda(AB)}{s_\lambda(I_N)} \delta_{\mu, \lambda}. \quad (82)$$

---

<sup>4</sup>If we replace the factor  $e^{\text{tr}(MM^*)}$  in the measure  $d\Omega^a$  by a hypergeometric tau function  $\tau_r(N, MM^*, I_N)$ , then the factor  $(N)_\lambda$  in the right hand side of (123) should be replaced by  $\frac{1}{r_\lambda(N)}$  [38].

Below  $\mathbf{p}_\infty = (1, 0, 0, \dots)$ .

$$\int_{\mathbb{C}^{n^2}} s_\lambda(AZBZ^+) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2 Z = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbf{p}_\infty)} \quad (83)$$

and

$$\int_{\mathbb{C}^{n^2}} s_\mu(AZ) s_\lambda(Z^+B) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2 Z = \frac{s_\lambda(AB)}{s_\lambda(\mathbf{p}_\infty)} \delta_{\mu,\lambda}. \quad (84)$$

We recall that

$$s_\lambda(I_N) = (N)_\lambda s_\lambda(\mathbf{p}_\infty), \quad \chi_\lambda(1) = |\lambda|! s_\lambda(\mathbf{p}_\infty)$$

Lemma 2 allows to pick up Hurwitz numbers from matrix integrals in many ways. Details may be found in the Appendix B Below we describe the simplest examples of integrals of tau function which are not tau function.

We use the simplest (the so-called vacuum) 2-KP tau function

$$\tau_1^{2\text{KP}}(X, \mathbf{p}) := \sum_{\lambda} s_\lambda(X) s_\lambda(\mathbf{p}) = e^{\text{tr}V(X, \mathbf{p})}, \quad \tau_1^{2\text{KP}}(X, \mathbf{p}_\infty) = e^{\text{tr}X} \quad (85)$$

where

$$V(z, \mathbf{p}) = \sum_{m>0} \frac{z^m p_m}{m}$$

$z$  may be a number and may be a matrix.

We use the simplest BKP tau function [31]

$$\tau_1^{\text{BKP}}(X) := \sum_{\lambda} s_\lambda(X) = \prod_{N>i>j} (1 - x_i x_j)^{-1} \prod_{i=1}^N (1 - x_i)^{-1} \quad (86)$$

Then we have

**Example.**  $\mathbb{CP}^1$  with three ramification point. In particular we have the following integrals generating Hurwitz numbers

$$\begin{aligned} & \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \frac{s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)}) s_\lambda(\mathbf{p}^{(3)})}{s_\lambda(\mathbf{p}_\infty)} = \\ &= \int e^{\text{tr}V(M_1 M_2, \mathbf{p}^{(3)})} \prod_{i=1,2} e^{\text{tr}V(M_i^*, \mathbf{p}^{(i)})} d\Omega^a(M_i, M_i^*) \\ &= \int e^{\text{tr}V(\Lambda M, \mathbf{p}^{(1)}) + \text{tr}V(M^*, \mathbf{p}^{(2)})} d\Omega^a(M, M^*), \quad p_m^{(3)} = \text{tr} \Lambda^m \end{aligned}$$

where  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ .

**Example.** An analog of (75) for  $\mathbb{RP}^2$  with three ramification points with two arbitrary profiles at 0 and at  $\infty$  with fixed length in the third point:

$$\begin{aligned} & \sum_{\lambda} \frac{s_\lambda(I_N) s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)})}{(s_\lambda(\mathbf{p}_\infty))^2} \\ &= \int \tau_1^{\text{BKP}}(M_1 M_2) \prod_{i=1,2} e^{V(\text{tr} M_i^*, \mathbf{p}^{(i)})} d\Omega^a(M_i, M_i^*) \quad a = \mathbb{C}, \mathbb{N}, \mathbb{H} \\ &= \int e^{\text{tr}(\Lambda M_1 M_2)} \tau_1^{\text{BKP}}(M_1^*) e^{\text{tr}V(M_2^*, \mathbf{p})} \prod_{i=1,2} d\Omega^{\mathbb{C}}(M_i, M_i^*), \quad p_m^{(2)} = \text{tr} \Lambda^m \end{aligned}$$

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## A Appendices

### A.1 Macdonald polynomials

This Appendix is a result of a discussion with John Harnad.

One can write down the scalar product where Macdonald polynomials  $P_\lambda(q^a, q; \mathbf{p})$  are orthonormal, by the integral over all power sums variables  $\mathbf{p}$  as follows

$$\langle f, g \rangle = \int_{\mathbb{C}^\infty} f(\mathbf{p})g(\mathbf{p}^*) \prod_{m=1}^{\infty} e^{-\frac{1}{m}|p_m|^2} \frac{p_m(a, q)}{2\pi i m} dp_m \wedge dp_m^* \quad (87)$$

where  $p_m^*$  is the complex conjugate to  $p_m$ . In this basis

$$\langle p_\Delta p_{\Delta'} \rangle = \frac{1}{w_\Delta(a, q)} \delta_{\Delta, \Delta'}, \quad w_\Delta(a, q) = \frac{p_\Delta(a, q)}{z_\Delta}$$

Exactly this ratio appears in the character expansion formula

$$s_\lambda(\mathbf{p}(a, q)) = \sum_{\Delta} \chi_\lambda(\Delta) w_\Delta(a, q) \quad (88)$$

Also one can write

$$\sum_{\Delta \in \mathbf{P}} \frac{p_\Delta(a, q)}{z_\Delta} P_\Delta(\mathbf{p}) P_\Delta(\bar{\mathbf{p}}) = e^{\sum_{m>0} \frac{1}{m} p_m \bar{p}_m p_m(a, q)} = \sum_{\Delta \in \mathbf{P}} \frac{p_\Delta(a, q)}{z_\Delta} \sum_{\lambda \in \mathbf{P}} s_\lambda(\mathbf{p}\bar{\mathbf{p}}) \chi_\lambda(\Delta) \quad (89)$$

where  $\mathbf{p}\bar{\mathbf{p}}$  denotes the set  $(p_1 \bar{p}_1, p_2 \bar{p}_2, \dots)$ .

Equation (49) corresponds to the case  $q \rightarrow 1$  where Macdonald polynomials convert to Jack ones.

The relation of of hypergeometric tau functions to the quantum integrable systems is unclear. The combinatorial and geometric interpretations of hypergeometric tau functions parametrized by pairs  $a_i, q_i$  in the TL case will be considered in [46].

### A.2 Hirota equations for the BKP tau function with two discrete time variables.

The BKP hierarchy we are interested in was introduced in [21]. It was used to construct various matrix models [40], [31], [41]. Hirota equations for the BKP hierarchy of Kac-van de Leur were presented in [21]. However in our case we need more general version which includes both discrete variables  $N$  and  $n$ , see [31]. The BKP tau function we need has the following form

$$\tau^{\text{BKP}}(N, n, \mathbf{p}|g) = \langle N + n | e^{\sum_{m>0} \frac{1}{m} \bar{p}_m J_m} g | n \rangle \quad (90)$$

where Clifford algebra element  $g$  may be considered as an element of  $\mathbb{O}(2\infty + 1)$  group which specifies the choice of the BKP tau function,

$$J_m = \sum_{i \in \mathbb{Z}} : \psi_i \psi_{i+m}^\dagger :$$

are Fourier modes of current operators, see details in [31]. Hirota equations for tau function (90) may be obtained by a certain specification of the Hirota equations for the two-sided BKP tau function

$$\tau^{\text{BKP}}(N, n, \mathbf{p}, \bar{\mathbf{p}}|g) = \langle N + n | e^{\sum_{m>0} \frac{1}{m} p_m J_m} g e^{\sum_{m>0} \frac{1}{m} p_m J_{-m}} | n \rangle$$

see [31], which in our notations are

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{N'+n'-N-n-2} e^{V(\mathbf{p}'-\mathbf{p}, z)} \tau(N' - 1, n', \mathbf{p}' - [z^{-1}], \bar{\mathbf{p}}') \tau(N + 1, n, \mathbf{p} + [z^{-1}], \bar{\mathbf{p}}) \\ & + \oint \frac{dz}{2\pi i} z^{N+n-N'-n'-2} e^{V(\mathbf{p}-\mathbf{p}', z)} \tau(N' + 1, n', \mathbf{p}' + [z^{-1}], \bar{\mathbf{p}}') \tau(N - 1, n, \mathbf{p} - [z^{-1}], \bar{\mathbf{p}}) \\ & = \oint \frac{dz}{2\pi i} z^{n'-n} e^{V(\bar{\mathbf{p}}'-\bar{\mathbf{p}}, z^{-1})} \tau(N' - 1, n' + 1, \mathbf{p}', \bar{\mathbf{p}}' - [z]) \tau(N + 1, n - 1, \mathbf{p}, \bar{\mathbf{p}} - [z]) \\ & + \oint \frac{dz}{2\pi i} z^{n-n'} e^{V(\bar{\mathbf{p}}'-\bar{\mathbf{p}}, z^{-1})} \tau(N' + 1, n' - 1, \mathbf{p}', \bar{\mathbf{p}}' + [z]) \tau(N - 1, n + 1, \mathbf{p}, \bar{\mathbf{p}} + [z]) \\ & + \frac{(-1)^{n'+n}}{2} (1 - (-1)^{N'+N}) \tau(N', n', \mathbf{p}', \bar{\mathbf{p}}') \tau(N, n, \mathbf{p}, \bar{\mathbf{p}}) \end{aligned} \quad (91)$$

see also [42]. Here  $\mathbf{p} = (p_1, p_2, \dots)$ ,  $\mathbf{p}' = (p'_1, p'_2, \dots)$ ,  $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \dots)$ ,  $\bar{\mathbf{p}}' = (\bar{p}'_1, \bar{p}'_2, \dots)$ , and

$$V(z, \mathbf{p}) = \sum_{m>0} \frac{1}{m} z^m p_m$$

The notation  $\mathbf{p} + [z^{-1}]$  denotes the set  $(p_1 + z^{-1}, p_2 + z^{-2}, p_3 + z^{-3}, \dots)$ .

**Remark 11.** Actually up to some simple factor the two-sided BKP tau function of [31] coincides with the two-component BKP tau function of [21] and Hirota equations (91) basically coincide with the Hirota equations for the two-component BKP, see Appendix in [42].

To obtain Hirota equations for (90) we chose  $\bar{\mathbf{p}} = \bar{\mathbf{p}}' = 0$ .

For  $n' = n + 1$ , we obtain (see [31])

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{N'-N-1} e^{V(\mathbf{p}'-\mathbf{p}, z)} \tau(N' - 1, n + 1, \mathbf{p}' - [z^{-1}]|g) \tau(N + 1, n, \mathbf{p} + [z^{-1}]|g) \\ & + \oint \frac{dz}{2\pi i} z^{N-N'-3} e^{V(\mathbf{p}-\mathbf{p}', z)} \tau(N' + 1, n + 1, \mathbf{p}' + [z^{-1}]|g) \tau(N - 1, n, \mathbf{p} - [z^{-1}]|g) \\ & = \tau(N' + 1, n, \mathbf{p}'|g) \tau(N - 1, n + 1, \mathbf{p}|g) - \frac{1}{2} (1 - (-1)^{N'+N}) \tau(N', n + 1, \mathbf{p}'|g) \tau(N, n, \mathbf{p}|g) \end{aligned} \quad (92)$$

For  $n' = n$ , we obtain Hirota equations as in [21]

$$\begin{aligned} & \oint \frac{dz}{2\pi i} z^{N'-N-2} e^{\xi(\mathbf{t}'-\mathbf{t}, z)} \tau(N' - 1, n, \mathbf{t}' - [z^{-1}]) \tau(N + 1, n, \mathbf{t} + [z^{-1}]) \\ & + \oint \frac{dz}{2\pi i} z^{N-N'-2} e^{\xi(\mathbf{t}-\mathbf{t}', z)} \tau(N' + 1, n, \mathbf{t}' + [z^{-1}]) \tau(N - 1, n, \mathbf{t} - [z^{-1}]) \\ & = \frac{1}{2} (1 - (-1)^{N'+N}) \tau(N', n, \mathbf{t}') \tau(N, n, \mathbf{t}) \end{aligned} \quad (93)$$

Let us write down some of them. Taking  $N' = N + 1$  and all  $p_i = p'_i$ ,  $i \neq 1$  in (92) and picking up the terms linear in  $p'_1 - p_1$  we obtain

$$\begin{aligned} & \frac{1}{2} \tau(N, n + 1, \mathbf{p}) \frac{\partial^2 \tau(N + 1, n, \mathbf{p})}{\partial^2 p_1} - \frac{1}{2} \frac{\tau(N, n + 1, \mathbf{p})}{\partial^2 p_1} \tau(N + 1, n, \mathbf{p}) = \\ & \frac{\partial \tau(N + 2, n, \mathbf{p})}{\partial p_1} \tau(N - 1, n + 1, \mathbf{p}) - \frac{\partial \tau(N + 1, n + 1, \mathbf{p})}{\partial p_1} \tau(N, n, \mathbf{p}) \end{aligned} \quad (94)$$

Taking  $N' = N + 1$  and all  $p_i = p'_i$ ,  $i \neq 2$  in (93) and picking up the terms linear in  $p'_2 - p_2$  we obtain

$$\begin{aligned} & \frac{1}{2} \frac{\partial \tau(N, n, \mathbf{p})}{\partial p_2} \tau(N + 1, n, \mathbf{p}) - \frac{1}{2} \tau(N, n, \mathbf{p}) \frac{\partial \tau(N + 1, n, \mathbf{p})}{\partial p_2} + \frac{1}{2} \frac{\partial^2 \tau(N, n, \mathbf{p})}{\partial^2 p_1} \tau(N + 1, n, \mathbf{p}) \\ & + \frac{1}{2} \tau(N, n, \mathbf{p}) \frac{\partial^2 \tau(N + 1, n, \mathbf{p})}{\partial^2 p_1} - \frac{\partial \tau(N, n, \mathbf{p})}{\partial p_1} \frac{\partial \tau(N + 1, n, \mathbf{p})}{\partial p_1} = \tau(N + 2, n, \mathbf{p}) \tau(N - 1, n, \mathbf{p}) \end{aligned} \quad (95)$$



### A.3 Fermionic formulae

Details may be found in [26, 31]. Let  $\{\psi_i, \psi_i^\dagger, i \in \mathbb{Z}\}$  are Fermi creation and annihilation operators that satisfy the usual anticommutation relations and vacuum annihilation conditions

$$[\psi_i^{(a)}, \psi_j^{\dagger(b)}]_+ = \delta_{ij} \delta_{a,b}, \quad \psi_i^{(1)} |n, * \rangle = \psi_{-i-1}^{\dagger(1)} |n, * \rangle = 0, \quad \psi_i^{(2)} |*, n \rangle = \psi_{-i-1}^{\dagger(2)} |*, n \rangle = 0 \quad \text{if } i < n, \quad (96)$$

Sometimes we will omit the superscript (1) in particular write  $\psi$  instead of  $\psi^{(1)}$ .

The hypergeometric tau functions may be written as follows

$$\tau_r^{\text{TL}}(n, \mathbf{p}, \bar{\mathbf{p}}) = g(n) \langle n | e^{\sum_{m>0} \frac{1}{m} J_m p_m} e^{\sum_{m>0} \frac{1}{m} p_m A_m} | n \rangle$$

where  $J_m = \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+m}^\dagger$  and  $A_m = \sum_{i \in \mathbb{Z}} r(i) \dots r(i-m) \psi_i \psi_{i-m}^\dagger$ . The semiinfinite TL may be described either putting by  $r(N) = 0$  or, it is may be suitable to present it in form

$$\tau_r^{\text{TL}}(M, n, \mathbf{p}, \bar{\mathbf{p}}) = (-1)^{\frac{M(M+1)}{2}} g(n) \langle M+n, -M-n | e^{\sum_{m>0} \frac{1}{m} J_m^2 p_m - \frac{1}{m} p_m A_m} e^{\sum_{n \in \mathbb{Z}} \psi_i^{(1)} \psi_{-i-1}^{\dagger(2)}} | n, -n \rangle$$

For BKP [21] one needs to introduce an additional Fermi mode  $\phi$  which anticommutes with each other Fermi operator except itself:  $\phi^2 = \frac{1}{2}$ , and  $\phi|0\rangle = \frac{1}{\sqrt{2}}|0\rangle$ . Then

$$\tau_r^{\text{BKP}}(N, n, \mathbf{p}, \bar{\mathbf{p}}) = g(n) \langle N+n | e^{\sum_{m>0} \frac{1}{m} B_m p_m} e^\omega | n \rangle = \langle N+n | e^{\sum_{m>0} \frac{1}{m} J_m p_m} e^{-\sum_{i \in \mathbb{Z}} U_i : \psi_i \psi_i^\dagger :} e^\omega | n \rangle \quad (97)$$

and

$$\tau_r^{\text{BKP}}(N = \infty, 0, \mathbf{p}) = \langle 0 | e^{\sum_{m>0} \frac{1}{m} B_m p_m} e^\omega e^{\omega^\dagger} | 0 \rangle = g(n) \sum_{\lambda \in \mathbf{P}} r_\lambda(0) s_\lambda(\mathbf{p}) \quad (98)$$

where

$$r(i) = e^{U_{i-1} - U_i} \quad (99)$$

and

$$\begin{aligned} \omega &= \sum_{i>j} \psi_i \psi_j - \sqrt{2} \phi \sum_{i \in \mathbb{Z}} \psi_i \\ \omega_- &= \sum_{i>j \geq 0} \psi_i \psi_j - \sqrt{2} \phi \sum_{i \geq 0} \psi_i, \quad \omega_+ = \sum_{i>j \geq 0} (-)^{i+j} \psi_{-j-1}^\dagger \psi_{-i-1}^\dagger + \sqrt{2} \phi \sum_{i \geq 0} \psi_{-i-1}^\dagger, \\ B_m &= \sum_{i \in \mathbb{Z}} \frac{1}{r(i)} \dots \frac{1}{r(i+m)} \psi_i \psi_{i+m}^\dagger \end{aligned}$$

and

$$g(n) = \langle n | e^{\sum_{i \in \mathbb{Z}} U_i : \psi_i \psi_i^\dagger :} | n \rangle =$$

$$e^{-U_0 + \dots - U_{n-1}} \quad \text{if } n > 0 \quad (100)$$

$$1 \quad \text{if } n = 0 \quad (101)$$

$$e^{U_{-1} + \dots + U_n} \quad \text{if } n < 0 \quad (102)$$

### A.4 BKP tau functions.

**Hirota equations for multicomponent BKP.** This is a particular case of the multicomponent BKP tau function, introduced in [21],

$$\tau(\mathbf{N}; \mathbf{s}) := \langle N^{(1)}, \dots, N^{(p)} | e^{\sum_{a=1}^p \sum_{i>0} \beta^a s^{(a)}} h^{(1, \dots, p)} | 0, 0 \rangle \quad (103)$$

where  $h^{(1, \dots, p)}$  solves

$$\left[ h^{(1, \dots, p)} \otimes h^{(1, \dots, p)}, \sum_{a=1}^p \sum_{i \in \mathbb{Z}} \psi_i^{(a)} \otimes \psi_i^{\dagger(a)} + \sum_{a=1,2} \sum_{i \in \mathbb{Z}} \psi_i^{\dagger(a)} \otimes \psi_i^{(a)} + \varphi \otimes \varphi \right] = 0 \quad (104)$$

From (104) the multicomponent BKP Hirota equations are obtained [21]:

$$\begin{aligned} & \sum_{a=1}^p \oint \frac{dz}{2\pi i} z^{N^{(a)'} - N^{(a)} - 2} e^{V(s^{(a)'} - s^{(a)}, z)} \tau \left( \mathbf{N}_{-}^{[a]'}; \mathbf{s}_{-}^{[a]'}(z) \right) \tau \left( \mathbf{N}_{+}^{[a]}; \mathbf{s}_{+}^{[a]}(z) \right) \\ & + \sum_{a=1}^p \oint \frac{dz}{2\pi i} z^{N^{(a)} - N^{(a)'} - 2} e^{V(s^{(a)} - s^{(a)'}, z)} \tau \left( \mathbf{N}_{+}^{[a]'}; \mathbf{s}_{+}^{[a]'}(z) \right) \tau \left( \mathbf{N}_{-}^{[a]}; \mathbf{s}_{-}^{[a]}(z) \right) \\ & = \frac{1}{2} (1 - (-1)^{\sum_{a=1}^p (N^{(a)'} + N^{(a)})}) \tau(\mathbf{N}'; \mathbf{s}') \tau(\mathbf{N}; \mathbf{s}) \end{aligned} \quad (105)$$

where

$$\begin{aligned} \mathbf{N}_{\pm}^{[a]} &:= (N^{(1)}, \dots, N^{(a-1)}, N^{(a)} \pm 1, N^{(a+1)}, \dots, N^{(p)}) \\ \mathbf{s}_{\pm}^{[a]}(z) &:= (s^{(1)}, \dots, s^{(a-1)}, s^{(a)} \pm [z^{-1}], s^{(a+1)}, \dots, s^{(p)}) \end{aligned}$$

In (105),  $\mathbf{N} = (N^{(1)}, \dots, N^{(p)})$  and  $\mathbf{N}' = (N^{(1)'}, \dots, N^{(p)'})$  are two independent sets of vacuum charges, while  $s^{(a)} = (s_1^{(a)}, s_2^{(a)}, s_3^{(a)})$  and  $s^{(a)'} = (s_1^{(a)'}, s_2^{(a)'}, s_3^{(a)'})$ ,  $a = 1, \dots, p$ , are two independent sets of the multicomponent BKP higher times.

**Pfaffian.** If  $A$  an anti-symmetric matrix of an odd order its determinant vanishes. For even order, say  $k$ , the following multilinear form in  $A_{ij}, i < j \leq k$

$$\text{Pf}[A] := \sum_{\sigma} \text{sgn}(\sigma) A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(k-1), \sigma(k)} \quad (106)$$

where sum runs over all permutation restricted by

$$\sigma : \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1), \quad (107)$$

coincides with the square root of  $\det A$  and is called the *Pfaffian* of  $A$ . As one can see the Pfaffian contains  $1 \cdot 3 \cdot 5 \cdots (k-1) =: (k-1)!!$  terms.

**BKP tau functions** [31]. A class of BKP tau functions has the following form

$$\tau^{\text{BKP}}(N, n, \mathbf{p}; A) = \sum_{h_1 > \cdots > h_N \geq 0} \bar{A}_h(n) s_{\{h\}}(\mathbf{p})$$

where  $s_{\{h\}} := s_{\lambda}$ ,  $h_i = \lambda_i - i + N$ ,  $i = 1, \dots, N$ . The factors  $\bar{A}_h(n)$  on the right-hand side are determined in terms a pair  $(A, a) =: \bar{A}$  where  $A$  is an infinite skew symmetric matrix and  $a$  an infinite vector. For a strict partition  $h = (h_1, \dots, h_N)$ , the numbers  $\bar{A}_h(n)$  are defined as the Pfaffian of an antisymmetric  $2k \times 2k$  matrix  $\tilde{A}$  as follows:

$$\bar{A}_h(n) := \text{Pf}[\tilde{A}] \quad (108)$$

where for  $N = 2k$  even

$$\tilde{A}_{ij} = -\tilde{A}_{ji} := A_{h_i+n, h_j+n}, \quad 1 \leq i < j \leq 2k \quad (109)$$

and for  $N = 2k-1$  odd

$$\tilde{A}_{ij}(n) = -\tilde{A}_{ji}(n) := \begin{cases} A_{h_i+n, h_j+n} & \text{if } 1 \leq i < j \leq 2k-1 \\ a_{h_i+n} & \text{if } 1 \leq i < j = 2k. \end{cases} \quad (110)$$

In addition we set  $\bar{A}_0 = 1$ .

The fermionic realization for this tau function is

$$\tau^{\text{BKP}}(N, n, \mathbf{p}; A) = \langle N + n | e^{\sum_{m>0} \frac{J_m p_m}{m}} e^{\sum_{i>j} A_{ij} \psi_i \psi_j + \sqrt{2} \sum_i a_i \psi_i \phi} | n \rangle$$

see [31].

**2KP tau functions** . A class of 2KP tau functions has the following form

$$\tau^{\text{2KP}}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}; B) = \sum_{\substack{h_1^{(1)} > \dots > h_N^{(1)} \geq 0 \\ h_1^{(2)} > \dots > h_N^{(2)} \geq 0}} B_{h^{(1)}, h^{(2)}}(n) s_{\{h^{(1)}\}}(\mathbf{p}^{(1)}) s_{\{h^{(2)}\}}(\mathbf{p}^{(2)})$$

The factors  $B_{h^{(1)}, h^{(2)}}(n)$  on the right-hand side is determined in terms an infinite matrix  $B$ . For a pair of strict partitions  $h^{(i)}$ ,  $i = 1, 2$ , the numbers  $B_{h^{(1)}, h^{(2)}}(n)$  are defined as follows:

$$B_{h^{(1)}, h^{(2)}}(n) := \det \left[ B_{h_i^{(1)} + n, h_j^{(2)} + n} \right]_{i, j=1, \dots, N} \quad (111)$$

compare to [34].

The fermionic realization for this tau function is

$$\tau^{\text{2KP}}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)}; B) = \langle N + n, -N + n | e^{\sum_{i=1,2} \sum_{m>0} \frac{J_m^{(i)} p_m^{(i)}}{m}} e^{\sum_{i,j} B_{ij} \psi_i^{(1)} \psi_{-1-j}^{(2)}} | n, n \rangle$$

For skew-symmetric  $B$  this tau function is the square of the DKP tau function [42].

**Determinants and pfaffians of special matrices.** We will look at determinants of degenerate matrices of form

$$A_{ij} = b_i c_j, \quad i, j = 1, \dots, n \quad (112)$$

where  $b_i$  and  $c_j$  are odd Grassmannian numbers. We see that

$$\det A = n! b_1 \cdots b_n c_n \cdots c_1 \quad (113)$$

For instance for a  $2 \times 2$  matrix (112) we have  $\det A = 2b_1 b_2 c_2 c_1$ .

Now, let  $A$  is a skew-symmetric matrix given by

$$A_{ij} = b_i c_j, i < j, \quad i, j = 1, \dots, 2n \quad (114)$$

Then for both  $b_i$  and  $c_i$  are odd Grassmannian numbers we get

$$\text{Pf } A = (2n - 1)!! b_1 \cdots c_{2n} b_{2n} \cdots b_1 \quad (115)$$

For both  $b_i$  and  $c_i$  are even Grassmannian numbers we obtain

$$\text{Pf } A = b_1 \cdots b_{2n} c_{2n} \cdots c_1 \quad (116)$$

**Exponentials.** Let  $\xi_i$  and  $\eta_i$  are odd Grassmannian numbers. We have

$$e^{\sum_{i,j} A_{ij} \xi_i \eta_j} = 1 + \sum_{k>0} A_{(\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)} \xi_{\alpha_1} \cdots \xi_{\alpha_k} \eta_{\beta_k} \cdots \eta_{\beta_1} \quad (117)$$

where

$$A_{(\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k)} = \det (A_{\alpha_i \beta_j})_{i, j=1, \dots, k} \quad (118)$$

Now let  $A$  is a skew-symmetric matrix (114) and  $\{a_i\}, i > 0$  is a set of (even) numbers.

**Quasi-tau functions** One can consider the following series in the Schur functions (compare to [16])

$$\tau^{[\text{E}]}(n, \{\mathbf{p}^i\}) := g(n) \sum_{\lambda \in \text{P}} r_\lambda(n) \prod_{i=1}^{\text{E}} s_\lambda(\mathbf{p}^{(i)}) \quad (119)$$

It may be presented in forms

$$\tau^{[\text{E}]}(n, \{\mathbf{p}^i\}) = g(n) \langle n + N | e^{\sum_{m>0} p_m^{(1)} B_m + \sum_{i=2}^k p_m^{(i)} J_m^{(i)}} e^{\omega^{[\text{E}]}} | 0 \rangle = \langle n + N | e^{\sum_{i=1}^k p_m^{(i)} J_m^{(i)}} e^{-\sum_{i \in \mathbb{Z}} U_i : \psi_i \psi_i^\dagger :} e^{\omega^{[\text{E}]}} | 0 \rangle \quad (120)$$

where

$$\omega^{[E]} := \sum_{i>j} \prod_{a=1}^E \left( \psi_i^{(a)} \psi_j^{(a)} \right) - \sqrt{2} \phi \sum_{i \in \mathbb{Z}} \prod_{a=1}^E \psi_i^{(a)} \quad (121)$$

which may be viewed as a sum of commutative elements in the tensor product  $\hat{o}(2\infty+1) \otimes \cdots \otimes \hat{o}(2\infty+1)$ .

Let us note that

$$e^{\omega^{[E]}} |n\rangle = 1 + \sum_{N>0} c_E \underbrace{|\lambda, N+n\rangle \otimes \cdots \otimes |\lambda, N+n\rangle}_E \quad (122)$$

where  $c_E = 1$  for even  $E$  and  $c_E = (2E-1)!!$  for  $E$  odd.

By coupling (122) with the vector

$$\langle N+n | e^{\sum_{a=1}^E \sum_{m>0} \frac{1}{m} J_m^{(a)} p_m^{(a)}} e^{-\sum_{i \in \mathbb{Z}} U_i : \psi_i \psi_i^\dagger :}$$

we obtain the right hand side of (119).

For  $E = 1$  we obtain hypergeometric  $\tau_r^{\text{BKP}}(N, n, \mathbf{p})$ . For  $E = 2$  we obtain  $\tau_r^{\text{TL}}(N, n, \mathbf{p}^{(1)}, \mathbf{p}^{(2)})$ .

## B Matrix integrals as generating functions of Hurwitz numbers and of Hurwitz generating series

The task of this section is to present Hurwitz generating series (5) in form of matrix integrals. The idea is that various integrals of products of the 2KP and BKP tau functions of matrix argument <sup>5</sup> result in the series (5).

Below we will write down matrix integrals which generate Hurwitz numbers themselves rather than weighted sums (as (75) does), though we also present some generating series for the sums.

**Notation. Useful relations.** We shall exploit the known formulae for integrations of Schur functions over the unitary group and over complex matrices. Earlier they were used in [32] to clarify some links between matrix models and integrable hierarchies.

As in [32] instead of power sums written by small bold characters (like  $\mathbf{p}$ ) sometimes where it is suitable we shall use the matrix arguments written by large character. Say,  $s_\lambda(X)$  and  $\tau(X)$  are respectively equal to  $s_\lambda(X) := s_\lambda(\mathbf{p}(X))$  and  $\tau(X) := \tau(\mathbf{p}(X))$  where  $\mathbf{p}(X) = (p_1(X), p_2(X), \dots)$ , where  $p_m(X) = \text{tr} X^m$ .

We use the following notations

- $d_* U$  is the normalized Haar measure on  $\mathbb{U}(N)$ :  $\int_{\mathbb{U}(N)} d_* U = 1$
- $Z$  is a complex matrix,  $Z = UX(1+J)U^\dagger$  (the Schur decomposition), where  $X = \text{diag}(z_i)$  is diagonal,  $J$  is strictly upper triangle,  $U \in \mathbb{U}(N)$

$$\begin{aligned} d\Omega^c(Z, Z^\dagger) &= \pi^{-n^2} e^{-\text{tr}(ZZ^\dagger)} \prod_{i,j=1}^N d\Re Z_{ij} d\Im Z_{ij} \\ &= c_Z d_* U \prod_{N \geq i > j} |z_i - z_j|^2 \prod_{i=1}^N e^{-|z_i|^2} d^2 z_i \left[ e^{-\text{tr} J J^\dagger} d^2 J_{ij} \right] \end{aligned}$$

where the part related to the upper triangular factor in brackets is not important for our problems.

- $M$  is a normal matrix,  $Z = UXU^\dagger$ , where  $X = \text{diag}(z_i)$  is diagonal,  $U \in \mathbb{U}(N)$

$$\begin{aligned} d\Omega^N(M, M^\dagger) &= \pi^{-n^2} e^{-\text{tr}(MM^\dagger)} \prod_{i,j=1}^N d\Re M_{ij} d\Im M_{ij} \\ &= c_M d_* U \prod_{N \geq i > j} |z_i - z_j|^2 \prod_{i=1}^N e^{-|z_i|^2} d^2 z_i \end{aligned}$$

<sup>5</sup> If  $\mathbf{p} = (p_1, p_2, \dots)$  and  $p_m = \text{tr} M^m$  where  $M$  is a matrix, we may write  $\tau = \tau(M) := \tau(\mathbf{p}(M))$  and call it tau function of matrix argument

- $H^{(1)}$  is a Hermitian matrix and  $H^{(2)}$  is anti-Hermitian one,  $H^{(c)} = U^{(c)} X^{(c)} U^{(c)\dagger}$ ,  $X^{(c)} = \text{diag}(x_i^{(c)})$ ,  $U, U^{(c)} \in \mathbb{U}(N)$ ,  $c = 1, 2$ . Measures

$$\begin{aligned}
d\Omega^{\text{H}\text{U}}(H^{(1)}, H^{(2)}, U) &= e^{-\text{tr}(H^{(1)} U H^{(2)} U^\dagger)} d_* U \prod_{i \leq j} d\Re H^{(1)} d\Im H^{(2)} \prod_{i < j} d\Im H^{(1)} d\Re H^{(2)}, \\
d\Omega^{\text{H}}(H^{(1)}, H^{(2)}) &= \int_{\mathbb{U}(N)} e^{-\text{tr}(H^{(1)} U H^{(2)} U^\dagger)} d_* U \prod_{i \leq j} d\Re H^{(1)} d\Im H^{(2)} \prod_{i < j} d\Im H^{(1)} d\Re H^{(2)} \\
&= c_{\text{H}} \prod_{c=1,2} d_* U^{(c)} \prod_{N \geq i > j} (x_i^{(c)} - x_j^{(c)}) \prod_{i=1}^N e^{-x_i^{(1)} x_i^{(2)}} dx_i^{(1)} dx_i^{(2)}
\end{aligned}$$

where the constants  $c_a$ ,  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ , are chosen for normalization:  $\int d\Omega_\rho^{(a)} = 1$ .

**Remark 12.** In what follows, for unification and to save space, we shall use the notation  $M$  and  $M^*$  replacing the pairs  $Z, Z^\dagger$ ,  $M, M^\dagger$  and also  $H^{(1)}, H^{(2)}$ . In the last case the matrices  $M$  and  $M^*$  are not related by the Hermitian conjugation.

These measures provides the relation

$$\int s_\lambda(M) s_\mu(M^*) d\Omega^a(M, M^*) = (N)_\lambda \delta_{\lambda, \mu} \quad (123)$$

where  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ . This relation was used in [38], [39], [32], [16], [36], for for models of Hermitian, complex and normal matrices.<sup>6</sup>

By  $I_N$  we shall denote the  $N \times N$  unit matrix. Then (for instance see [30])

**Lemma 3.** *Let  $A$  and  $B$  be normal matrices (i.e. matrices diagonalizable by unitary transformations). Then*

$$\int_{\mathbb{U}(N)} s_\lambda(AUBU^{-1}) d_* U = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(I_N)}, \quad (124)$$

For  $A, B \in GL(N)$  we have

$$\int_{\mathbb{U}(n)} s_\mu(AU) s_\lambda(U^{-1}B) d_* U = \frac{s_\lambda(AB)}{s_\lambda(I_N)} \delta_{\mu, \lambda}. \quad (125)$$

Below  $\mathbf{p}_\infty = (1, 0, 0, \dots)$ .

$$\int_{\mathbb{C}^{n^2}} s_\lambda(AZBZ^+) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2 Z = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(\mathbf{p}_\infty)} \quad (126)$$

and

$$\int_{\mathbb{C}^{n^2}} s_\mu(AZ) s_\lambda(Z^+B) e^{-\text{Tr}ZZ^+} \prod_{i,j=1}^n d^2 Z = \frac{s_\lambda(AB)}{s_\lambda(\mathbf{p}_\infty)} \delta_{\mu, \lambda}. \quad (127)$$

We recall that

$$s_\lambda(I_N) = (N)_\lambda s_\lambda(\mathbf{p}_\infty), \quad \chi_\lambda(1) = |\lambda|! s_\lambda(\mathbf{p}_\infty)$$

**Remark 13.** Usually the relation (126) is written down for positive matrices  $A$  and  $B$ . Equation (126) may be derived using the Gauss integration. Let us note that for any  $A, B \in GL(N)$  the Gauss integrals of products of type  $\prod_i (\text{tr} C^{k_i})^{m_i}$  where  $C = AZBZ^\dagger$  yields sums of terms  $\prod_i (\text{tr} A^{k'_i})^{m'_i} \prod_i (\text{tr} A^{k''_i})^{m''_i}$  which depend only on the spectrums of matrices  $A$  and  $B$ .

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<sup>6</sup>If we replace the factor  $e^{\text{tr}(MM^*)}$  in the measure  $d\Omega^a$  by a hypergeometric tau function  $\tau_r(N, MM^*, I_N)$ , then the factor  $(N)_\lambda$  in the right hand side of (123) should be replaced by  $\frac{1}{r_\lambda(N)}$  [38].

Lemma 3 allows to pick up Hurwitz numbers from matrix integrals in many ways. Below we describe a set of the most natural ones.

First of all, step by step applying (126) we arrive at

$$\int_{[C^{N^2}]^{\times(F-1)}} s_\lambda \left( A_F \left( Z_{F-1} A_{F-1} Z_{F-1}^\dagger \cdots Z_1 A_1 Z_1^\dagger \right) \right) \prod_{i=1}^{F-1} d\Omega^C(Z_i, Z_i^\dagger) = \frac{\prod_{i=1}^F s_\lambda(A_i)}{(s_\lambda(\mathbf{p}_\infty))^{F-1}} \quad (128)$$

We obtain the multi-matrix analogues of the Itsykson-Zuber integral

$$\int e^{\text{tr} V(X_{F-1} M_1 X_1 M_1^* \cdots M_{F-2} X_{F-2} M_{F-2}^*, \mathbf{p})} \prod_{i=1}^{F-1} d\Omega^C(M_i, M_i^*) = (s_\lambda(\mathbf{p}_\infty))^2 \frac{s_\lambda(\mathbf{p})}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^{F-1} \frac{s_\lambda(X_i)}{s_\lambda(\mathbf{p}_\infty)} \quad (129)$$

The relation (129) gives the generating function for Hurwitz numbers on the sphere with  $F$  ramification points (compare to [50]).

Actually we use the simplest (the so-called vacuum) 2-KP tau function

$$\tau_1^{2\text{KP}}(X, \mathbf{p}) := \sum_{\lambda} s_\lambda(X) s_\lambda(\mathbf{p}) = e^{\text{tr} V(X, \mathbf{p})}, \quad \tau_1^{2\text{KP}}(X, \mathbf{p}_\infty) = e^{\text{tr} X} \quad (130)$$

where

$$V(z, \mathbf{p}) = \sum_{m>0} \frac{z^m p_m}{m}$$

( $z$  may be a number and may be a matrix) as the integrand in (129). To get analogues of (129) for the projective plane we use the simplest BKP tau function [31]

$$\tau_1^{\text{BKP}}(X) := \sum_{\lambda} s_\lambda(X) = \prod_{N>i>j} (1 - x_i x_j)^{-1} \prod_{i=1}^N (1 - x_i)^{-1} \quad (131)$$

Then we have

$$\int_{C^{N^2 \times (F-1)}} \tau_1^{\text{BKP}} \left( X_F Z_1 X_1 Z_1^\dagger \cdots Z_{F-1} X_{F-1} Z_{F-1}^\dagger \right) \prod_{i=1}^{F-1} e^{-\text{tr} Z_i Z_i^\dagger} d^2 Z_i = s_\lambda(\mathbf{p}_\infty) \prod_{i=1}^F \frac{s_\lambda(X_i)}{s_\lambda(\mathbf{p}_\infty)} \quad (132)$$

The last formula is the generating function for the Hurwitz numbers for projective plane with  $F$  ramification points.

**Example.**  $\mathbb{CP}^1$  with three ramification point. In particular we have the following integrals generating Hurwitz numbers

$$\begin{aligned} & \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \frac{s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)}) s_\lambda(\mathbf{p}^{(3)})}{s_\lambda(\mathbf{p}_\infty)} = \\ &= \int e^{\text{tr} V(M_1 M_2, \mathbf{p}^{(3)})} \prod_{i=1,2} e^{\text{tr} V(M_i^*, \mathbf{p}^{(i)})} d\Omega^a(M_i, M_i^*) \\ &= \int e^{\text{tr} V(\Lambda M, \mathbf{p}^{(1)}) + \text{tr} V(M^*, \mathbf{p}^{(2)})} d\Omega^a(M, M^*), \quad p_m^{(3)} = \text{tr} \Lambda^m \\ &= \int e^{\text{tr} V(X_2 M_1 X_1 M_1^*, \mathbf{p})} \prod_{i=1,2} d\Omega^C(M_i, M_i^*), \quad p_m^{(i)} = \text{tr}(X_i)^m, \quad i = 1, 2 \\ &= \int e^{\text{tr}(X_3 M_1 X_1 M_1^* M_2 X_2 M_2^*)} \prod_{i=1}^3 d\Omega^C(M_i, M_i^*), \quad p_m^{(i)} = \text{tr}(X_i)^m, \quad i = 1, 2, 3 \end{aligned}$$

where  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ .

**Example.** Coverings of  $\mathbb{RP}^2$  with three ramification point. To get a generating integral of the related Hurwitz numbers one can take a  $\mathbb{CP}^1$  Hurwitz integral with 4 ramification points and replace any of the 2KP tau functions under the integral by a BKP tau function. For instance,

$$\frac{\prod_{i=1}^4 s_\lambda(\mathbf{p}^{(i)})}{(s_\lambda(\mathbf{p}_\infty))^2} = \int e^{\text{tr}V(\Lambda M_1^* M_2^*, \mathbf{p}^{(3)})} \prod_{i=1,2} e^{\text{tr}V(M_i, \mathbf{p}^{(i)})} d\Omega^{\mathbb{C}}(M_i, M_i^*), \quad p_m^{(4)} = \text{tr} \Lambda^m$$

Then the generating integral for  $\mathbb{RP}^2$  Hurwitz numbers for covering given by 3 ramification points may be written as

$$\begin{aligned} & \frac{\prod_{i=1}^3 s_\lambda(\mathbf{p}^{(i)})}{(s_\lambda(\mathbf{p}_\infty))^2} = \\ &= \int \tau_1^{\text{BKP}}(\Lambda M_1^* M_2^*) \prod_{i=1,2} e^{\text{tr}V(M_i, \mathbf{p}^{(i)})} d\Omega^{\mathbb{C}}(M_i, M_i^*), \quad p_m^{(3)} = \text{tr} \Lambda^m \\ &= \int e^{\text{tr}V(\Lambda M_1^* M_2^*, \mathbf{p}^{(2)})} e^{\text{tr}V(M_1, \mathbf{p}^{(1)})} \tau_1^{\text{BKP}}(M_2) \prod_{i=1,2} d\Omega^{\mathbb{C}}(M_i, M_i^*), \quad p_m^{(3)} = \text{tr} \Lambda^m \end{aligned}$$

**Example.** An analog of (75) for  $\mathbb{RP}^2$  with three ramification points with two arbitrary profiles at 0 and at  $\infty$  with fixed length in the third point:

$$\begin{aligned} & \sum_{\lambda} \frac{s_\lambda(I_N) s_\lambda(\mathbf{p}^{(1)}) s_\lambda(\mathbf{p}^{(2)})}{(s_\lambda(\mathbf{p}_\infty))^2} \\ &= \int \tau_1^{\text{BKP}}(M_1 M_2) \prod_{i=1,2} e^{V(\text{tr} M_i^*, \mathbf{p}^{(i)})} d\Omega^a(M_i, M_i^*) \\ &= \int e^{\text{tr}(\Lambda M_1 M_2)} \tau_1^{\text{BKP}}(M_1^*) e^{\text{tr}V(M_2^*, \mathbf{p})} \prod_{i=1,2} d\Omega^{\mathbb{C}}(M_i, M_i^*), \quad p_m^{(2)} = \text{tr} \Lambda^m \\ &= \int e^{\text{tr}(M_1 M_2 M_3)} \tau_1^{\text{BKP}}(M_3^*) e^{\text{tr}V(M_1^*, \mathbf{p}^{(1)}) + \text{tr}V(M_2^*, \mathbf{p}^{(2)})} \prod_{i=1}^3 d\Omega^{\mathbb{C}}(M_i, M_i^*) \\ &= \int e^{\text{tr}V(M_1 M_2 M_3, \mathbf{p}^{(1)})} \tau_1^{\text{BKP}}(M_3^*) e^{\text{tr}(M_1^* + \text{tr}V(M_2^*, \mathbf{p}^{(2)}))} \prod_{i=1}^3 d\Omega^{\mathbb{C}}(M_i, M_i^*) \end{aligned}$$

where  $a = \mathbb{C}, \mathbb{N}, \mathbb{H}$ .

We also need the following relation which follows applying (126) and the applying (125):

$$\int_{[\mathbb{U}(N)]^{\times G}} \int_{[\mathbb{C}^{N^2}]^{\times G}} s_\lambda(Y_G) \prod_{i=1}^G d_* U_i \prod_{i=1}^{2G} d\Omega^{\mathbb{C}}(M_i, M_i^*) = (s_\lambda(\mathbf{p}_\infty))^{-2G} \quad (133)$$

where

$$\begin{aligned} Y_G &= (M_{2G} U_G M_{2G}^* M_{2G-1} U_G^\dagger M_{2G-1}^*) \cdots (M_2 U_1 M_2^* M_1 U_1^\dagger M_1^*) \\ &= \int s_\lambda(\Lambda M_1^* \cdots M_G^*) \prod_{i=1}^G \tau_1^{\text{BKP}}(M_i) d\Omega^{\mathbb{C}}(M_i, M_i^*) = \frac{s_\lambda(\Lambda)}{s_\lambda(\mathbf{p}_\infty)^G} \end{aligned} \quad (134)$$

We get

Let  $M_1, \dots, M_G$ ,  $G > 0$ , be  $N \times N$  normal matrices, that is they may be presented as  $M_i = U_i X_i U_i^{-1}$  where  $U_i \in \mathbb{U}(N)$  and  $(X_i)_{ab} = x_a^{(i)} \delta_{a,b}$ , with  $x_a^{(i)} \in \mathbb{C}$ ,  $a = 1, \dots, N$ , being the eigenvalues of  $M_i$ . Let  $\Lambda$  be diagonal.

$$\tau_r^{\text{E}, \text{F}+G} \left( N, n, \Lambda, \bar{\mathbf{p}}^{(1)}, \dots, \bar{\mathbf{p}}^{(\text{F}-1)}, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(G)} \right) := \quad (135)$$

$$\sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} r_\lambda(n) (s_\lambda(\mathbf{p}_\infty))^E \frac{s_\lambda(\Lambda)}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^{F-1} \frac{s_\lambda(\bar{\mathbf{p}}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^G \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} = \quad (136)$$

$$\frac{1}{c} \int \tau_r^{\mathbf{E}, \mathbf{F}} \left( N, n, \Lambda M_1^* \cdots M_G^*, \bar{\mathbf{p}}^{(1)}, \dots, \bar{\mathbf{p}}^{(F-1)} \right) \prod_{i=1}^G e^{\text{tr} V(M_i, \mathbf{p}^{(i)})} d\Omega_\rho^\alpha(M_i, M_i^*) \quad (137)$$

where  $\alpha = \mathbf{N}, \mathbf{C}, \mathbf{H}$  i.e. each  $M_i$  and is respectively normal, complex matrix

where  $c = \int d\mu_\rho(M, M^\dagger)$ , and where  $\rho$  is rather arbitrary which provides  $c$  to be finite, and which defines the relation between functions  $r$  and  $\tilde{r}$  as follows:

$$n! \tilde{r}(1) \cdots \tilde{r}(n) = r(1) \cdots r(n) \int_{z \geq 0} z^n e^{-z} \rho(z) dz \quad (138)$$

In particular, for  $\rho = 1$ ,  $\tilde{r} = r$ .

Proof follows from the following relations:

$$\begin{aligned} e^{\text{tr} V(M_i, \mathbf{p}^{(i)})} &= \sum_{\lambda} s_\lambda(M_i) s_\lambda(\mathbf{p}^{(i)}) , \\ \tau_r^{\mathbf{E}, \mathbf{F}} \left( N, n, \Lambda M_1^\dagger \cdots M_G^\dagger, \bar{\mathbf{p}}^{(1)}, \dots, \bar{\mathbf{p}}^{(F-1)} \right) &= \\ \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} r_\lambda(n) (s_\lambda(\mathbf{p}_\infty))^2 \frac{s_\lambda(\Lambda M_1^\dagger \cdots M_{F-1}^\dagger)}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^{F-1} \frac{s_\lambda(\bar{\mathbf{p}}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \end{aligned}$$

then, having in mind that  $s_\lambda(UMU^\dagger) = s_\lambda(M)$ , from (138) which results in

$$r_\lambda(N) \int s_\lambda(M) s_\mu(M^\dagger) d\mu(M, M^\dagger) = \tilde{r}_\lambda(N) (N)_\lambda \delta_{\lambda, \mu} , \quad (139)$$

used in [36]<sup>7</sup>.

For  $\tau_r^{\mathbf{E}, \mathbf{F}}$  we take the simplest (the so-called, vacuum) TL tau function, we obtain

The generating function for the Hurwitz numbers on the sphere with  $F$  ramification points may be written as follows:

$$\tau^{2, \mathbf{F}} \left( N, \Lambda, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F-1)} \right) := \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} (s_\lambda(\mathbf{p}_\infty))^2 \frac{s_\lambda(M)}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^{F-1} \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} \quad (140)$$

$$= \int e^{\text{tr} V(\Lambda M_1^* \cdots M_{F-2}^*, \mathbf{p}^{(F-1)})} \prod_{i=1}^{F-2} e^{\text{tr} V(M_i, \mathbf{p}^{(i)})} d\Omega^a(M_i, M_i^*) \quad (141)$$

For independent proof one may use

$$e^{\text{tr} V(\Lambda M_1^\dagger \cdots M_{F-2}^*, \mathbf{p}^{(F-1)})} = \sum_{\lambda} s_\lambda(\Lambda M_1^* \cdots M_{F-1}^\dagger) s_\lambda(\mathbf{p}^{(F-1)}) ,$$

The generating function for the Hurwitz numbers on the projective plane with  $F$  ramification points may be written as follows:

$$\begin{aligned} \tau^{1, \mathbf{F}} \left( N, \Lambda, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F-1)} \right) &:= \sum_{\substack{\lambda \in \mathbf{P} \\ \ell(\lambda) \leq N}} s_\lambda(\mathbf{p}_\infty) \frac{s_\lambda(\Lambda)}{s_\lambda(\mathbf{p}_\infty)} \prod_{i=1}^{F-1} \frac{s_\lambda(\mathbf{p}^{(i)})}{s_\lambda(\mathbf{p}_\infty)} = \\ \int \tau_1^{\text{BKP}}(X) \prod_{i=1}^{F-1} e^{\text{tr} V(M_i, \mathbf{p}^{(i)})} d\mu(M_i, M_i^\dagger) &= \end{aligned} \quad (142)$$

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<sup>7</sup>see [44] where this was used to define a deformed scalar product.



$$\int \tau_1^{2\text{KP}} \left( X, \mathbf{p}^{(F-1)} \right) \tau_1^{\text{BKP}} \left( M_{F-2} \right) d\mu \left( M_{F-1}, M_{F-1}^\dagger \right) \prod_{i=1}^{F-2} e^{\text{tr } V(M_i, \mathbf{p}^{(i)})} d\mu \left( M_i, M_i^\dagger \right) \quad (143)$$

where

$$X = \Lambda M_1^\dagger \cdots M_{F-2}^\dagger$$

and

$$\tau_1^{\text{BKP}}(X) := \sum_{\lambda} s_{\lambda}(X) = \prod_{N > i > j} (1 - x_i x_j)^{-1} \prod_{i=1}^N (1 - x_i)^{-1}$$

Obtained relations allows to change F keeping the Euler characteristic E. Now, we want to change E. This is even easier and may be done in a straightforward way

$$\tau^{E-2, F-2} \left( N, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F-2)} \right) = \int_{\mathbb{U}(N)} \tau^{E, F} \left( N, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(F-2)}, U, U^\dagger \right) d_* U \quad (144)$$

Now, we can construct generating functions for  $H_{d, N}^E$  where for even E we start from a Toda lattice tau function and for E odd we start from a BKP tau function.

$$d\Omega(Z, Z^\dagger) = e^{-\text{tr } Z_i Z_i^\dagger} d^2 Z_i$$

$$\sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} (s_{\lambda}(\mathbf{p}_{\infty}))^{2-2G} \frac{s_{\lambda}(\mathbf{p})}{s_{\lambda}(\mathbf{p}_{\infty})} \prod_{i=1}^F \frac{s_{\lambda}(\Lambda_i)}{s_{\lambda}(\mathbf{p}_{\infty})} = \int \tau_1^{2\text{KP}} (X_F Y_G, \mathbf{p}) \prod_{i=1}^F d\Omega(Z_i, Z_i^\dagger) \prod_{i=1}^G d_* U_i \prod_{i=1}^{2G} d\Omega(C_i, C_i^\dagger) = \quad (145)$$

$$\int \tau_1^{2\text{KP}} (\tilde{X}_F \tilde{Y}_G, \mathbf{p}) \prod_{i=1}^F d_* U_i \prod_{i=1}^G d\Omega(Z_i, Z_i^\dagger) \prod_{i=1}^{2G} \prod_{i=1}^F d_* W_i DDXSSWSW \quad (146)$$

and

$$\begin{aligned} \int \tau_1^{\text{BKP}} (X_F Y_G) \prod_{i=1}^F e^{-\text{tr } Z_i Z_i^\dagger} d^2 Z_i \prod_{i=1}^G d_* U_i \prod_{i=1}^{2G} e^{-\text{tr } C_i C_i^\dagger} d^2 C_i &= \\ &= \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} (s_{\lambda}(\mathbf{p}_{\infty}))^{1-2G} \prod_{i=1}^F \frac{s_{\lambda}(\Lambda_i)}{s_{\lambda}(\mathbf{p}_{\infty})} \end{aligned} \quad (147)$$

where

$$\begin{aligned} X_F &= \Lambda_F \left( Z_{F-1} \Lambda_{F-1} Z_{F-1}^\dagger \cdots Z_1 \Lambda_1 Z_1^\dagger \right) \\ Y_G &= \left( C_{2G} U_G C_{2G}^\dagger C_{2G-1} U_G^\dagger C_{2G-1}^\dagger \right) \cdots \left( C_2 U_1 C_2^\dagger C_1 U_1^\dagger C_1^\dagger \right) \end{aligned}$$

and  $Z_i$  and  $C_i$  are complex matrices.

$$\begin{aligned} \int \tau_1^{2\text{KP}} (X_F Y_G, \mathbf{p}) \prod_{i=1}^F e^{-\text{tr } Z_i Z_i^\dagger} d^2 Z_i \prod_{i=1}^G d_* U_i \prod_{i=1}^{2G} e^{-\text{tr } C_i C_i^\dagger} d^2 C_i &= \\ &= \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} (s_{\lambda}(\mathbf{p}_{\infty}))^{2-2G} \frac{s_{\lambda}(\mathbf{p})}{s_{\lambda}(\mathbf{p}_{\infty})} \prod_{i=1}^F \frac{s_{\lambda}(\Lambda_i)}{s_{\lambda}(\mathbf{p}_{\infty})} \end{aligned} \quad (148)$$

and

$$\begin{aligned} \int \tau_1^{\text{BKP}} (X_F Y_G) \prod_{i=1}^F d\Omega^a(M_i, M_i^*) \prod_{i=1}^G d_* U_i \prod_{i=1}^{2G} d\Omega^c(Z, Z^\dagger) &= \\ &= \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} (s_{\lambda}(\mathbf{p}_{\infty}))^{1-2G} \prod_{i=1}^F \frac{s_{\lambda}(\Lambda_i)}{s_{\lambda}(\mathbf{p}_{\infty})} \end{aligned} \quad (149)$$

where

$$X_F = \Lambda_F \left( U_{F-1} M_{F-1} U_{F-1}^\dagger \cdots U_1 M_1 U_1^\dagger \right), \quad Y_G = \left( Z_{2G} U_G Z_{2G}^\dagger Z_{2G-1} U_G^\dagger Z_{2G-1}^\dagger \right) \cdots \left( Z_2 U_1 Z_2^\dagger Z_1 U_1^\dagger Z_1^\dagger \right)$$

and  $M_i, M_i^*$  may be conjugated complex matrices, conjugated normal ones, or a pair of two nonrelated matrices, one being Hermitian matrix while the other is an anti-Hermitian one,  $U_i \in \mathbb{U}$  and  $Z_i$  are complex matrices.

Then we use the known formula (for instance see [30])

$$\int_{\mathbb{U}(N)} s_\lambda(AU B U^\dagger) dU = \frac{s_\lambda(A) s_\lambda(B)}{s_\lambda(I_N)} \quad (150)$$

where  $I_N$  is the unit matrix (see for instance [30]) and Cauchy-Littlewood formula

$$e^{\sum_{m>0} \frac{1}{m} p_m p_m^*} = \sum_{\lambda} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{p}^*) \quad (151)$$

where

$$p_m^* = \text{tr}(AUBU^\dagger)^m \quad (152)$$

like it was done in [43]. Writing the product of  $K$  matrices as  $A_1 B_1$  where  $B_1 = A_2 \cdots A_K$  diagonalizing, then repeating  $K-1$  times we obtain

$$I = \text{Vol} \mathbb{U}(N) \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \frac{s_\lambda(\mathbf{p}) \prod_{i=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \quad (153)$$

which may be related to more complicated Hurwitz numbers with  $K+1$  arbitrary profiles, namely to the sums  $S_{\mathbb{CP}^1}(d|b|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(K+1)})$ .

Similarly, we can integrate hypergeometric  $\tau_r^{TL}(\mathbf{p}, A_1 \cdots A_K)$  and  $\tau_r^{BKP} A_1 \cdots A_K$  (instead of the simplest TL tau function given by Itsykson-Zuber  $e^{\text{tr} AUBU^\dagger}$ ).

We obtain

The generating function  $S_{\mathbb{CP}^1}(d|b|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(K+1)})$  is constructed as the following matrix integral

$$\int_{\mathbb{U}(N) \times \cdots \times \mathbb{U}(N)} \tau_r^{TL}(n, \mathbf{p}, U_1 A_1 U_1^\dagger U_2 A_2 U_2^\dagger \cdots U_K A_K U_K^\dagger) \prod_{i=1}^K dU_i = \quad (154)$$

$$\sum_B (q e^{\beta n})^d \frac{\beta^b}{b!} \frac{\prod_{i=1}^k (a_i + n)^{l_i}}{\prod_{i=1}^s b_i^d (b_i + n)^{l_i^*}} S_{\mathbb{CP}^1}^N(B) \prod_{i=1}^{K+1} \mathbf{p}_{\Delta^{(i)}}^{(i)} \quad (155)$$

where the sum is taken by all  $B = (d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(K+1)})$  and where  $\mathbf{p}^{(i)} = (p_1^{(i)}, p_2^{(i)}, \dots)$  and

$$p_m^{(i)} = \text{tr} A_i^m, \quad i = 1, \dots, K, \quad p_m^{(K+1)} = p_m$$

From (150)-(152) we obtain

$$\int_{\mathbb{U}(N) \times \cdots \times \mathbb{U}(N)} \tau_r^{TL}(\mathbf{p}, U_1 A_1 U_1^\dagger U_2 A_2 U_2^\dagger \cdots U_K A_K U_K^\dagger) \prod_{i=1}^K dU_i = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_\lambda(n) \frac{s_\lambda(\mathbf{p}) \prod_{i=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \quad (156)$$

Then we apply the same steps as the Theorem 1.

Construct now generating function for  $S_{\mathbb{RP}^2}^N(d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(K)})$ . Put

$$\int_{\mathbb{U}(N) \times \cdots \times \mathbb{U}(N)} \tau_r^{BKP}(n, U_1 A_1 U_1^\dagger U_2 A_2 U_2^\dagger \cdots U_K A_K U_K^\dagger) \prod_{i=1}^K dU_i = \quad (157)$$

$$\sum_B (qe^{\beta n})^d \frac{\beta^b}{b!} \frac{\prod_{i=1}^s b_i^{-d} (b_i + n)^{l_i^*}}{\prod_{i=1}^k a_i^{-d} (a_i + n)^{l_i}} S_{\mathbb{RP}^2}^N(B) \prod_{i=1}^K \mathbf{p}_\Delta^{(i)} \quad (158)$$

where the sum is taken by all  $B = (d|l_1, \dots, l_k|l_1^*, \dots, l_s^*|\Delta^{(1)}, \dots, \Delta^{(K)})$  and where  $\mathbf{p}_i = (p_1^{(i)}, p_2^{(i)}, \dots)$  and

$$p_m^{(i)} = \text{tr} A_i^m, \quad i = 1, \dots, K$$

From (150)-(152) we obtain

$$\int_{\mathbb{U}(N) \times \dots \times \mathbb{U}(N)} \tau_r^{\text{BKP}}(U_1 A_1 U_1^\dagger U_2 A_2 U_2^\dagger \dots U_K A_K U_K^\dagger) \prod_{i=1}^K dU_i = \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} r_\lambda(n) \frac{\prod_{i=1}^K s_\lambda(A_i)}{(s_\lambda(I_N))^{K-1}} \quad (159)$$

Then we apply the same steps as the Theorem 1.

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